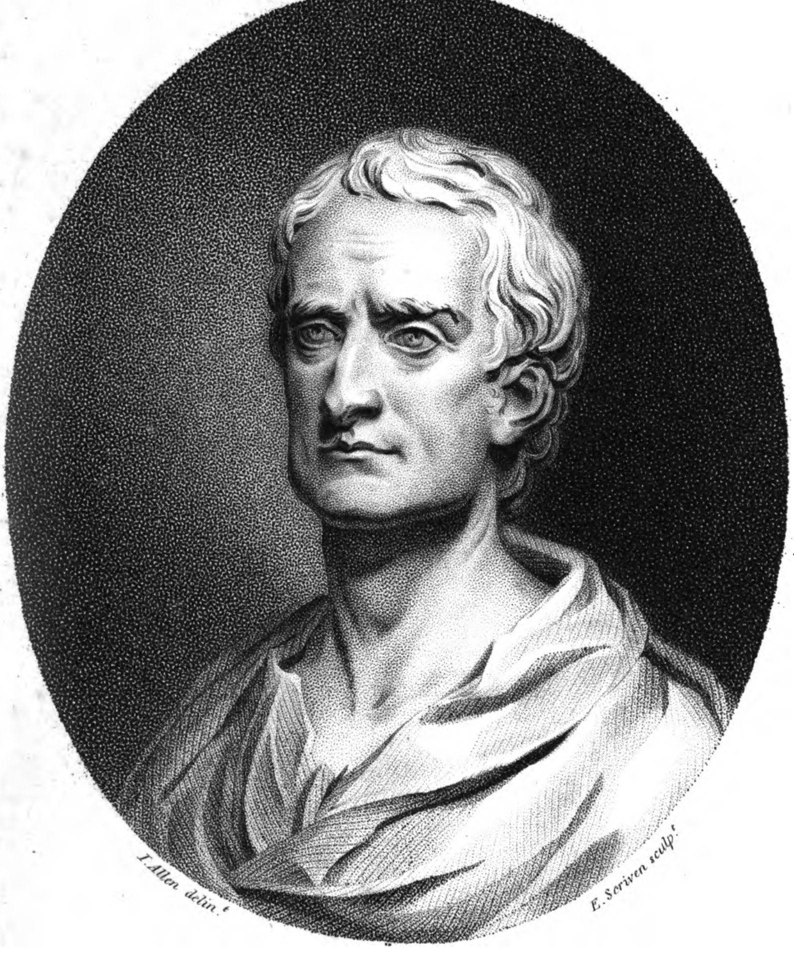

SIR ISAAC NEWTON.

From a Bust at the Royal Observatory, Greenwich.

London. Published Nov. 20. 1802. by H.D. Symonds Paternoster Row.

THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY.

BY

SIR ISAAC NEWTON.

Translated into English

BY ANDREW MOTTE.

TO WHICH ARE ADDED,

Newton's System of the World;

A SHORT

Comment on, and Defence of, the Principia,

BY W. EMERSON.

WITH

THE LAWS OF THE MOON'S MOTION

According to Gravity.

BY JOHN MACHIN,

Astron., Prof. at Gresh., and Sec. to the Roy. Soc.

A new Edition,

(With the LIFE of the AUTHOR; and a PORTRAIT, taken from the Bust in the Royal Observatory at Greenwich)

CAREFULLY REVISED AND CORRECTED BY

W. DAVIS,

Author of the "Treatise on Land Surveying," the "Use of the Globes," Editor of the "Mathematical Companion," &c. &c. &c.

IN THREE VOLUMES.

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TO THE

REV. NEVIL MASKELYNE,

D.D., F.R.S.

AND

ASTRONOMER ROYAL,

THIS EDITION

of

Sir ISAAC NEWTON's PRINCIPIA,

AND OF

THE SYSTEM OF THE WORLD,

&c. &c. &c.

IS

(With Permission)

Most humbly dedicated,

BY

HIS VERY OBEDIENT,

AND

MUCH OBLIGED SERVANT,

William Davis.

TO THE READER.

THE inconvenience arifing from the great fcarcity of former editions of Sir Ifaac Newton's PRINCIPIA, and SYSTEM OF THE WORLD, added to the exorbitant prices charged for them when to be met with, determined the Editor to undertake a New Edition of thofe Works; and he is impreffed with confidence, that no other apology will be thought neceffary, at a time when Mathematics is become a fafhionable fcience, and is looked upon as a neceffary acquifition in the polite world.

In compliance with the folicitations of feveral refpectable Mathematicians, to this Edition is added Mr. W. Emerfon's much admired COMMENT, and DEFENCE of Sir Ifaac Newton's PRINCIPIA, thereby rendering the Work more eafy of comprehenfion to ftudents, and others, not fully acquainted with the higher branches of Mathematics; and the Editor is not without hope that the whole will be found generally correct:----PERFECTION he has not yet thought of afpiring to.

W. DAVIS.

London, January,
1803.

TO

SIR HANS SLOANE, BART.

President

OF THE

College of Physicians,

AND OF

THE ROYAL SOCIETY.

SIR,

THE generous zeal you always ſhew for whatever tends to the progreſs and advancement of Learning, both demands and receives the univerſal acknowledgments of all who profeſs or value its ſeveral branches.

They juſtly admire, that, amidſt a cloſe attendance on the cares of your profeſſion, in which you now fill the moſt honourable ſeat, you are indefatigably promoting the improvement of natural knowledge, by carrying on ſome laudable deſigns of your own, by aſſiſting and encouraging others, and by adding new ſtores to that immenſe treaſure, already brought into your extenſive collection, of whatever is rare and valuable in nature or art.

Your beneficent diſpoſition to countenance and favour Science and Literature has procured you the eſteem of the Learned over all the world; and has induced a body of men, the moſt eminent for their ſkill and diligence in all uſeful enquiries, and

in purſuing diſcoveries for the public good, to make choice of you, to ſupply the place of him whoſe name will be an everlaſting honour to our age and nation.

To whom, therefore, but to you, ſhould I offer to inſcribe the tranſlation of the moſt celebrated Work of your illuſtrious predeceſſor? which, on account of its incomparable author, and from the dignity of the ſubject, claims and deſerves your acceptance, even though it paſſed through my hands: a leſs valuable piece I ſhould not have preſumed to preſent you with. I am, with the greateſt reſpect,

SIR,

Your moſt obedient, and

Moſt humble ſervant,

Andrew Motte.

THE

AUTHOR'S PREFACE.

SINCE the antients (as we are told by *Pappus*) made great account of the ſcience of mechanics in the inveſtigation of natural things; and the moderns, laying aſide ſubſtantial forms and occult qualities, have endeavoured to ſubject the phænomena of nature to the laws of mathematics, I have in this treatiſe cultivated mathematics ſo far as it regards philoſophy. The antients conſidered mechanics in a twofold reſpect; as rational, which proceeds accurately by demonſtration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But, as artificers do not work with perfect accuracy, it comes to paſs that mechanics is ſo diſtinguiſhed from geometry, that what is perfectly accurate is called geometrical; what is leſs ſo, is called mechanical. But the errors are not in the art, but in the artificers. He that works with leſs accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the moſt perfect mechanic of all; for the deſcription of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw theſe lines, but requires them to be drawn; for it requires that the learner ſhould firſt be taught to deſcribe theſe accurately, before he enters upon geometry; then it ſhews how by theſe operations problems may be ſolved. To deſcribe right lines and circles are problems, but not geometrical problems. The ſolution of theſe problems is required from mechanics; and by geometry the uſe of them, when ſo ſolved, is ſhewn; and it is the glory of geometry that from thoſe few principles, fetched from without, it is able to produce ſo many things. Therefore geometry is founded in mechanical practice, and is nothing but that part of univerſal mechanics which accurately propoſes and demonſtrates the art of meaſuring. But ſince the manual arts are chiefly

converſant in the moving of bodies, it comes to paſs that geometry is commonly referred to their magnitudes, and mechanics to their motion. In this ſenſe rational mechanics will be the ſcience of motions reſulting from any forces whatſoever, and of the forces required to produce any motions, accurately propoſed and demonſtrated. This part of mechanics was cultivated by the antients in the five powers which relate to manual arts, who conſidered gravity (it not being a manual power) no otherwiſe than as it moved weights by thoſe powers. Our deſign not reſpecting arts, but philoſophy, and our ſubject not manual but natural powers, we conſider chiefly thoſe things which relate to gravity, levity, elaſtic force, the reſiſtance of fluids, and the like forces, whether attractive or impulſive; and therefore we offer this work as mathematical principles of philoſophy; for all the difficulty of philoſophy ſeems to conſiſt in this—from the phænomena of motions to inveſtigate the forces of nature, and then from theſe forces to demonſtrate the other phænomena; and to this end the general propoſitions in the firſt and ſecond book are directed. In the third book we give an example of this in the explication of the Syſtem of the World; for by the propoſitions mathematically demonſtrated in the firſt book, we there derive from the celeſtial phænomena the forces of gravity with which bodies tend to the ſun and the ſeveral planets. Then from theſe forces, by other propoſitions which are alſo mathematical, we deduce the motions of the planets, the comets, the moon, and the ſea. I wiſh we could derive the reſt of the phænomena of nature by the ſame kind of reaſoning from mechanical principles; for I am induced by many reaſons to ſuſpect that they may all depend upon certain forces by which the particles of bodies, by ſome cauſes hitherto unknown, are either mutually impelled towards each other, and cohere in regular figures, or are repelled and recede from each other; which forces being unknown, philoſophers have hitherto attempted the ſearch of nature in vain; but I hope the principles here laid down will afford ſome light either to that or ſome truer method of philoſophy.

In the publication of this Work the moſt acute and univerſally learned Mr. *Edmund Halley* not only aſſiſted me with his pains in correcting the preſs and taking care of the ſchemes, but it was to his ſolicitations that its becoming public is owing; for when he had obtained of me my demonſtrations of the figure of the celeſtial orbits, he continually preſſed me to communicate the ſame to the *Royal Society*, who afterwards, by their kind encouragement and entreaties, engaged me to think of publiſhing them. But after I had begun to conſider the inequalities of the lunar motions, and had entered upon ſome other things relating to the laws and meaſures of gravity, and other forces; and the figures that would be deſcribed by bodies attracted according to given laws; and the motion of ſeveral bodies moving among themſelves; the motion of bodies in reſiſting mediums; the forces, denſities, and motions, of mediums; the orbits of the comets, and ſuch like; I put off that publication till I had made a ſearch into thoſe matters, and could put out the whole together. What relates to the lunar motions (being imperfect) I have put all together in the corollaries of prop. 66, to avoid being obliged to propoſe and diſtinctly demonſtrate the ſeveral things there contained in a method more prolix than the ſubject deſerved, and interrupt the ſeries of the ſeveral propoſitions. Some things, found out after the reſt, I choſe to inſert in places leſs ſuitable, rather than change the number of the propoſitions and the citations. I heartily beg that what I have here done may be read with candour; and that the defects I have been guilty of upon this difficult ſubject may be not ſo much reprehended as kindly ſupplied, and inveſtigated by new endeavours of my readers.

ISAAC NEWTON.

Cambridge, Trinity College,
May 8, 1686.

In the ſecond edition the ſecond ſection of the firſt book was enlarged. In the ſeventh ſection of the ſecond book

the theory of the refiftances of fluids was more accurately inveftigated, and confirmed by new experiments. In the third book the moon's theory and the præceffion of the equinoxes were more fully deduced from their principles; and the theory of the comets was confirmed by more examples of the calculation of their orbits, done alfo with greater accuracy.

In this third edition the refiftance of mediums is fomewhat more largely handled than before; and new experiments of the refiftance of heavy bodies falling in air are added. In the third book, the argument to prove that the moon is retained in its orbit by the force of gravity is enlarged on; and there are added new obfervations of Mr. *Pound*'s of the proportion of the diameters of *Jupiter* to each other: there are, befides, added Mr. *Kirk*'s obfervations of the comet in 1680; the orbit of that comet computed in an ellipfis by Dr. *Halley*; and the orbit of the comet in 1723, computed by Mr. *Bradley*.

THE

PREFACE

OF

MR. ROGER COTES,

TO THE

SECOND EDITION OF THIS WORK,

SO FAR AS IT RELATES TO THE INVENTIONS AND DISCOVERIES THEREIN CONTAINED.

THOSE who have treated of natural philofophy may be nearly reduced to three claffes. Of thefe, fome have attributed to the feveral fpecies of things fpecific and occult qualities, on which, in a manner unknown, they make the operations of the feveral bodies to depend. The fum of the doctrine of the fchools derived from *Ariftotle* and the Peripatetics is herein contained. They affirm that the feveral effects of bodies arife from the particular natures of thofe bodies; but whence it is that bodies derive thofe natures they do not tell us, and therefore they tell us nothing. And being entirely employed in giving names to things, and not in fearching into things themfelves, we may fay, that they have invented a philofophical way of fpeaking, but not that they have made known to us true philofophy.

Others, therefore, by laying afide that ufelefs heap of words, thought to employ their pains to better purpofe. Thefe fuppofed all matter homogeneous, and that the variety of forms which is feen in bodies arifes from fome very plain and fimple affections of the component particles; and by going on from fimple things to thofe which are more compounded, they certainly proceed right, if they attribute no other properties to thofe primary affections of the particles than nature has done. But when they take a liberty of imagining at pleafure unknown figures and magnitudes, and uncertain fituations

and motions of the parts; and, moreover, of ſuppoſing occult fluids, freely pervading the pores of bodies, endued with an all-performing ſubtilty, and agitated with occult motions; they now run out into dreams and chimeras, and neglect the true conſtitution of things; which certainly is not to be expected from fallacious conjectures, when we can ſcarcely reach it by the moſt certain obſervations. Thoſe who fetch from hypotheſes the foundation on which they build their ſpeculations, may form, indeed, an ingenious romance; but a romance it will ſtill be.

There is left, then, the third claſs, which profeſs experimental philoſophy. Theſe, indeed, derive the cauſes of all things from the moſt ſimple principles poſſible; but, then, they aſſume nothing as a principle that is not proved by phænomena. They frame no hypotheſes, nor receive them into philoſophy otherwiſe than as queſtions whoſe truth may be diſputed. They proceed, therefore, in a twofold method, ſynthetical and analytical. From ſome ſelect phænomena they deduce by analyſis the forces of nature, and the more ſimple laws of forces; and from thence by ſyntheſis ſhew the conſtitution of the reſt. This is that incomparably beſt way of philoſophizing which our renowned author moſt juſtly embraced before the reſt, and thought alone worthy to be cultivated and adorned by his excellent labours. Of this he has given us a moſt illuſtrious example by the explication of the Syſtem of the World, moſt happily deduced from the theory of gravity. That the virtue of gravity was found in all bodies, others ſuſpected or imagined before him; but he was the only and the firſt philoſopher that could demonſtrate it from appearances, and make it a ſolid foundation to the moſt noble ſpeculations.

I know, indeed, that ſome perſons, and thoſe of great name, too much prepoſſeſſed with certain prejudices, are unwilling to aſſent to this new principle, and are ready to prefer uncertain notions to certain. It is not my intention to detract from the reputation of theſe eminent men; I ſhall only lay before the reader ſuch conſiderations as will enable him to paſs an equitable ſentence in this diſpute.

Therefore, that we may begin our reafoning from what is moft fimple and neareft to us, let us confider a little what is the nature of gravity with us on earth, that we may proceed the more fafely when we come to confider it in the heavenly bodies that lie at fo vaft a diftance from us. It is now agreed by all philofophers, that all circumterreftrial bodies gravitate towards the earth. That no bodies really light are to be found, is now confirmed by manifold experience. That which is relative levity is not true levity, but apparent only; and arifes from the preponderating gravity of the contiguous bodies.

Moreover, as all bodies gravitate towards the earth, fo does the earth again towards bodies. That the action of gravity is mutual, and equal on both fides, is thus proved. Let the mafs of the earth be diftinguifhed into any two parts whatever, either equal, or any how unequal: now, if the weights of the parts towards each other were not mutually equal, the leffer weight would give way to the greater, and the two parts joined together would move on *ad infinitum* in a right line towards that part to which the greater weight tends; altogether againft experience. Therefore, we muft fay; that the weights of the parts are conftituted in equilibrio; that is, that the action of gravity is mutual and equal on both fides.

The weights of bodies, at equal diftances from the centre of the earth, are as the quantities of matter in the bodies. This is collected from the equal acceleration of all bodies that fall from a ftate of reft by the force of their weights; for the forces by which unequal bodies are equally accelerated muft be proportional to the quantities of the matter to be moved. Now, that all bodies are in falling equally accelerated, appears from hence,---that when the refiftance of the air is taken away, as it is under an exhaufted receiver, bodies falling defcribe equal fpaces in equal times; and this is yet more accurately proved by the experiments of pendulums.

The attractive forces of bodies at equal diftances are as the quantities of matter in the bodies; for fince bodies gravitate towards the earth, and the earth again towards bodies with equal moments, the weight of the earth towards every body,

or the force with which the body attracts the earth, will be equal to the weight of the same body towards the earth. But this weight was shewn to be as the quantity of matter in the body; and therefore the force with which every body attracts the earth, or the absolute force of the body, will be as the same quantity of matter.

Therefore the attractive force of the entire bodies arises from, and is compounded of, the attractive forces of the parts; because, as was just shewn, if the bulk of the matter be augmented or diminished, its virtue is proportionably augmented or diminished. We must, therefore, conclude that the action of the earth is compounded of the united actions of its parts; and therefore that all terrestrial bodies must attract each other mutually, with absolute forces that are as the matter attracting. This is the nature of gravity upon earth: let us now see what it is in the heavens.

That every body perseveres in its state either of rest, or of moving uniformly in a right line, unless in so far as it is compelled to change that state by forces impressed, is a law of nature universally received by all philosophers. But from thence it follows, that bodies which move in curve lines, and are therefore continually going off from the right lines that are tangents to their orbits, are by some continued force retained in those curvilinear paths. Since, then, the planets move in curvilinear orbits, there must be some force operating, by whose repeated actions they are perpetually made to deflect from the tangents.

Now it is collected by mathematical reasoning, and evidently demonstrated, that all bodies that move in any curve line described in a plane, and which, by a radius drawn to any point, whether quiescent, or any how moved, describe areas about that point proportional to the times, are urged by forces directed towards that point. This must, therefore, be granted. Since, then, all astronomers agree that the primary planets describe about the sun, and the secondary about the primary, areas proportional to the times, it follows that the forces by which they are perpetually turned aside from the rectilinear tangents, and made to revolve in curvilinear orbits, are directed towards the bodies that are situate in the

centres of the orbits. This force may, therefore, not improperly be called centripetal, in refpect of the revolving body; and in refpect of the central body, attractive; whatever caufe it may be imagined to arife from.

But, befides, thefe things muft be alfo granted as being mathematically demonftrated. If feveral bodies revolve with an equable motion in concentric circles, and the fquares of the periodic times are as the cubes of the diftances from the common centre, the centripetal forces will be reciprocally as the fquares of the diftances. Or, if bodies revolve in orbits that are very near to circles, and the apfides of the orbits reft, the centripetal forces of the revolving bodies will be reciprocally as the fquares of the diftances. That both thefe cafes hold in all the planets, all aftronomers confent. Therefore the centripetal forces of all the planets are reciprocally as the fquares of the diftances from the centres of their orbits. If any fhould object that the apfides of the planets, and efpecially of the moon, are not perfectly at reft, but are carried with a flow kind of motion *in confequentia*, one may give this anfwer;---that, though we fhould grant this very flow motion to arife from hence, that the proportion of the centripetal force is a little different from the duplicate, yet that we are able to compute mathematically the quantity of that aberration, and find it perfectly infenfible. For the ratio of the lunar centripetal force itfelf, which muft be the moft irregular of them all, will be, indeed, a little greater than the duplicate, but will be near fixty times nearer to that than it is to the triplicate. But we may give a truer anfwer, by faying, that this progreffion of the apfides arifes not from an aberration from the duplicate proportion, but from a quite different caufe, as is moft admirably fhewn in this philofophy. It is certain, then, that the centripetal forces with which the primary planets tend to the fun, and the fecondary to their primary, are accurately as the fquares of the diftances reciprocally.

From what has been hitherto faid, it is plain that the planets are retained in their orbits by fome force perpetually acting upon them; it is plain that that force is always direct-

ed towards the centres of their orbits; it is plain that its efficacy is augmented with the nearneſs to the centre, and diminiſhed with the ſame; and that it is augmented in the ſame proportion with which the ſquare of the diſtance is diminiſhed, and diminiſhed in the ſame proportion with which the ſquare of the diſtance is augmented. Let us now ſee whether, by making a comparifon between the centripetal forces of the planets and the force of gravity, we may not by chance find them to be of the ſame kind. Now they will be of the ſame kind, if we find on both ſides the ſame laws, and the ſame affections. Let us, then, firſt conſider the centripetal force of the moon, which is neareſt to us.

The rectilinear ſpaces, which bodies let fall from reſt deſcribe in a given time at the very beginning of the motion, when the bodies are urged by any forces whatſoever, are proportional to the forces. This appears from mathematical reaſoning. Therefore the centripetal force of the moon revolving in its orbit is to the force of gravity at the ſurface of the earth, as the ſpace which, in a very ſmall particle of time, the moon, deprived of all its circular force, and deſcending by its centripetal force towards the earth, would deſcribe, is to the ſpace which a heavy body would deſcribe, when falling by the force of its gravity near to the earth, in the ſame given particle of time. The firſt of theſe ſpaces is equal to the verſed ſine of the arc deſcribed by the moon in the ſame time, becauſe that verſed ſine meaſures the tranſlation of the moon from the tangent, produced by the centripetal force; and therefore may be computed, if the periodic time of the moon and its diſtance from the centre of the earth are given. The laſt ſpace is found by experiments of pendulums, as Mr. *Huygens* has ſhewn. Therefore, by making a calculation, we ſhall find that the firſt ſpace is to the latter; or the centripetal force of the moon revolving in its orbit will be to the force of gravity at the ſuperficies of the earth as the ſquare of the ſemi-diameter of the earth to the ſquare of the ſemi-diameter of the orbit. But, by what was ſhewn before, the very ſame ratio holds between the centripetal force of the moon revolving in its orbit, and the

centripetal force of the moon near the ſurface of the earth. Therefore the centripetal force near the ſurface of the earth is equal to the force of gravity. Therefore theſe are not two different forces, but one and the ſame; for if they were different, theſe forces united would cauſe bodies to deſcend to the earth with twice the velocity they would fall with by the force of gravity alone. Therefore it is plain that the centripetal force, by which the moon is perpetually either impelled or attracted out of the tangent and retained in its orbit, is the very force of terreſtrial gravity reaching up to the moon. And it is very reaſonable to believe that virtue ſhould extend itſelf to vaſt diſtances, ſince upon the tops of the higheſt mountains we find no ſenſible diminution of it. Therefore the moon gravitates towards the earth; but, on the other hand, the earth by a mutual action equally gravitates towards the moon; which is alſo abundantly confirmed in this philoſophy, where the tides in the ſea and the præceſſion of the equinoxes are treated of, which ariſe from the action both of the moon and of the ſun upon the earth. Hence, laſtly, we diſcover by what law the force of gravity decreaſes at great diſtances from the earth; for ſince gravity is no ways different from the moon's centripetal force, and this is reciprocally proportional to the ſquare of the diſtance, it follows that it is in that very ratio that the force of gravity decreaſes.

Let us now go on to the reſt of the planets. Becauſe the revolutions of the primary planets about the ſun, and of the ſecondary about Jupiter and Saturn, are phænomena of the ſame kind with the revolution of the moon about the earth; and becauſe it has been moreover demonſtrated that the centripetal forces of the primary planets are directed towards the centre of the ſun, and thoſe of the ſecondary towards the centres of Jupiter and Saturn, in the ſame manner as the centripetal force of the moon is directed towards the centre of the earth; and ſince, beſides, all theſe forces are reciprocally as the ſquares of the diſtances from the centres, in the ſame manner as the centripetal force of the moon is as the ſquare of the diſtance from the earth; we muſt of courſe conclude

that the nature of all is the ſame. Therefore as the moon gravitates towards the earth, and the earth again towards the moon, ſo alſo all the ſecondary planets will gravitate towards their primary, and the primary planets again towards their ſecondary; and ſo all the primary towards the ſun; and the ſun again towards the primary.

Therefore the ſun gravitates towards all the planets, and all the planets towards the ſun; for the ſecondary planets, while they accompany the primary, revolve the mean while with the primary about the ſun. Therefore, by the ſame argument, the planets of both kinds gravitate towards the ſun, and the ſun towards them. That the ſecondary planets gravitate towards the ſun is moreover abundantly clear from the inequalities of the moon; a moſt accurate theory of which, laid open with a moſt admirable ſagacity, we find explained in the third book of this Work.

That the attractive virtue of the ſun is propagated on all ſides to prodigious diſtances, and is diffuſed to every part of the wide ſpace that ſurrounds it, is moſt evidently ſhewn by the motion of the comets, which, coming from places immenſely diſtant from the ſun, approach very near to it; and ſometimes ſo near, that in their perihelia they almoſt touch its body. The theory of theſe bodies was altogether unknown to aſtronomers, till in our own times our excellent author moſt happily diſcovered it, and demonſtrated the truth of it by moſt certain obſervations; ſo that it is now apparent that the comets move in conic ſections having their foci in the ſun's centre, and by radii drawn to the ſun deſcribe areas proportional to the times. But from theſe phænomena it is manifeſt, and mathematically demonſtrated, that thoſe forces, by which the comets are retained in their orbits, reſpect the ſun, and are reciprocally proportional to the ſquares of the diſtances from its centre. Therefore the comets gravitate towards the ſun; and therefore the attractive force of the ſun not only acts on the bodies of the planets placed at given diſtances, and very nearly in the ſame plane, but reaches alſo to the comets in the moſt different parts of the heavens, and at the moſt different diſtances. This, therefore, is the nature of gravitating bodies to propagate their force

at all diſtances to all other gravitating bodies. But from thence it follows that all the planets and comets attract each other mutually, and gravitate mutually towards each other; which is alſo confirmed by the perturbation of Jupiter and Saturn, obſerved by aſtronomers, which is cauſed by the mutual actions of theſe two planets upon each other; as alſo from that very ſlow motion of the apſides above taken notice of, and which ariſes from a like cauſe.

We have now proceeded ſo far as to ſhew that it muſt be acknowledged that the ſun, and the earth, and all the heavenly bodies attending the ſun, attract each other mutually. Therefore all the leaſt particles of matter in every one muſt have their ſeveral attractive forces, whoſe effect is as their quantity of matter; as was ſhewn above of the terreſtrial particles. At different diſtances theſe forces will be alſo in the duplicate ratio of the diſtances reciprocally; for it is mathematically demonſtrated that particles attracting according to this law will compoſe globes attracting according to the ſame law.

The foregoing concluſions are grounded on this axiom, which is received by all philoſophers; namely, that effects of the ſame kind, that is, whoſe known properties are the ſame, take their riſe from the ſame cauſes, and have the ſame unknown properties alſo; for who doubts, if gravity be the cauſe of the deſcent of a ſtone in *Europe*, but that it is alſo the cauſe of the ſame deſcent in *America*? If there is a mutual gravitation between a ſtone and the earth in *Europe*, who will deny the ſame to be mutual in *America*? If in *Europe* the attractive force of a ſtone and the earth is compounded of the attractive forces of the parts, who will deny the like compoſition in *America*? If in *Europe* the attraction of the earth be propagated to all kinds of bodies, and to all diſtances, why may it not as well be propagated in like manner in *America*? All philoſophy is founded on this rule; for if that be taken away, we can affirm nothing of univerſals. The conſtitution of particular things is known by obſervations and experiments; and when that is done, it is by this rule

that we judge univerfally of the nature of fuch things in general.

Since, then, all bodies, whether upon earth or in the heavens, are heavy, fo far as we can make any experiments or obfervations concerning them, we muft certainly allow that gravity is found in all bodies univerfally; and in like manner, as we ought not to fuppofe that any bodies can be otherwife than extended, moveable, or impenetrable, fo we ought not to conceive that any bodies can be otherwife than heavy. The extenfion, mobility, and impenetrability of bodies become known to us only by experiments; and in the very fame manner their gravity becomes known to us. All bodies we can make any obfervations upon are extended, moveable, and impenetrable; and thence we conclude all bodies, and thofe we have no obfervations concerning, to be extended, and moveable, and impenetrable. So all bodies we can make obfervations on we find to be heavy; and thence we conclude all bodies, and thofe we have no obfervations of, to be heavy alfo. If any one fhould fay that the bodies of the fixed ftars are not heavy, becaufe their gravity is not yet obferved, they may fay, for the fame reafon, that they are neither extended, nor moveable, nor impenetrable, becaufe thefe affections of the fixed ftars are not yet obferved. In fhort, either gravity muft have a place among the primary qualities of all bodies, or extenfion, mobility, and impenetrability, muft not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extenfion, mobility, and impenetrability.

Some, I know, difapprove this conclufion, and mutter fomething about occult qualities. They are continually cavilling with us, that gravity is an occult property; and occult caufes are to be quite banifhed from philofophy. But to this the anfwer is eafy: that thofe are, indeed, occult caufes whofe exiftence is occult; and imagined, but not proved; but not thofe whofe real exiftence is clearly demonftrated by obfervations. Therefore gravity can by no means be called an occult caufe of the celeftial motions, becaufe it is plain from the phænomena that fuch a virtue does really exift. Thofe rather have re-

course to occult causes who set imaginary vortices, of a matter entirely fictitious, and imperceptible by our senses, to direct those motions.

But shall gravity be therefore called an occult cause, and thrown out of philosophy, because the cause of gravity is occult, and not yet discovered? Those who affirm this should be careful not to fall into an absurdity that may overturn the foundations of all philosophy; for causes use to proceed in a continued chain from those that are more compounded to those that are more simple: when we are arrived at the most simple cause, we can go no farther. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause were not the most simple. The most simple causes will you, then, call occult, and reject them? Then you must reject those that immediately depend upon them, and those which depend upon these last, till philosophy is quite cleared and disencumbered of all causes.

Some there are who say that gravity is præternatural, and call it a perpetual miracle; therefore they would have it rejected, because præternatural causes have no place in physics. It is hardly worth while to spend time in answering this ridiculous objection, which overturns all philosophy; for either they will deny gravity to be in bodies, which cannot be said, or else they will therefore call it præternatural, because it is not produced by the other affections of bodies, and therefore not by mechanical causes. But certainly there are primary affections of bodies; and these, because they are primary, have no dependance on the others. Let them consider whether all these are not in like manner præternatural, and in like manner to be rejected; and then what kind of philosophy we are like to have.

Some there are who dislike this celestial physics, because it contradicts the opinions of *Descartes*, and seems hardly to be reconciled with them. Let these enjoy their own opinion; but let them act fairly, and not deny the same liberty to us which they demand for themselves. Since the *Newtonian* Philosophy appears true to us, let us have the liberty to embrace and

retain it, and to follow causes proved by phænomena, rather than causes only imagined, and not yet proved. The business of true philosophy is to derive the natures of things from causes truly existent; and to enquire after those laws on which the Great Creator actually chose to found this most beautiful frame of the world; not those by which he might have done the same, had he so pleased. It is reasonable enough to suppose that, from several causes somewhat differing from each other, the same effect may arise; but the true cause will be that from which it truly and actually does arise: the others have no place in true philosophy. The same motion of the hour-hand in a clock may be occasioned either by a weight hung or a spring shut up within; but if a certain clock should be really moved with a weight, we should laugh at a man that would suppose it moved by a spring, and from that principle, suddenly taken up without farther examination, should go about to explain the motion of the index; for certainly the way he ought to have taken should have been actually to look into the inward parts of the machine, that he might find the true principle of the proposed motion. The like judgment ought to be made of those philosophers who will have the heavens to be filled with a most subtile matter, which is perpetually carried round in vortices; for if they could explain the phænomena ever so accurately by their hypotheses, we could not yet say that they have discovered true philosophy, and the true causes of the celestial motions, unless they could either demonstrate that those causes do actually exist, or, at least, that no other do exist. Therefore if it be made clear that the attraction of all bodies is a property actually existing *in rerum naturâ*, and if it be also shewn how the motions of the celestial bodies may be solved by that property, it would be very impertinent for any one to object that these motions ought to be accounted for by vortices, even though we should ever so much allow such an explication of those motions to be possible. But we allow no such thing; for the phænomena can by no means be accounted for by vortices, as our author has abundantly proved from the clearest reasons. So that men must be strangely fond of chimeras who can spend their

time ſo idly as in patching up a ridiculous figment, and ſetting it off with new comments of their own.

If the bodies of the planets and comets are carried round the ſun in vortices, the bodies ſo carried, and the parts of the vortices next ſurrounding them, muſt be carried with the ſame velocity and the ſame direction, and have the ſame denſity, and the ſame *vis inertiæ* anſwering to the bulk of the matter. But it is certain the planets and comets, when in the very ſame parts of the heavens, are carried with various velocities and various directions; therefore it neceſſarily follows that thoſe parts of the celeſtial fluid which are at the ſame diſtances from the ſun muſt revolve at the ſame time with different velocities in different directions; for one kind of velocity and direction is required for the motion of the planets, and another for that of the comets. But, ſince this cannot be accounted for, we muſt either ſay that all the celeſtial bodies are not carried about by vortices, or elſe that their motions are derived not from one and the ſame vortex, but from ſeveral diſtinct ones, which fill and pervade the ſpaces round about the ſun.

But if ſeveral vortices are contained in the ſame ſpace, and are ſuppoſed to penetrate each other, and to revolve with different motions, then, becauſe theſe motions muſt agree with thoſe of the bodies carried about by them, which are perfectly regular, and performed in conic ſections which are ſometimes very eccentric, and ſometimes nearly circles, one may reaſonably aſk, how it comes to paſs that theſe vortices remain entire, and have ſuffered no manner of perturbation in ſo many ages from the actions of the conflicting matter? Certainly, if theſe fictitious motions are more compounded and more hard to be accounted for than the true motions of the planets and comets, it ſeems to no purpoſe to admit them into philoſophy, ſince every cauſe ought to be more ſimple than its effect. Allowing men to indulge their own fancies, ſuppoſe any man ſhould affirm that the planets and comets are ſurrounded with atmoſpheres like our earth, which hypotheſis ſeems more reaſonable than that of vortices. Let him then affirm that theſe atmoſpheres, by their own nature, move about the ſun, and deſcribe conic

sections, which motion is much more easily conceived than that of the vortices penetrating each other. Lastly, that the planets and comets are carried about the sun by these atmospheres of their's; and then applaud his own sagacity in discovering the causes of the celestial motions. He that rejects this fable, must also reject the other; for two drops of water are not more like than this hypothesis of atmospheres, and that of votices.

Galileo has shewn, that when a stone projected moves in a parabola, its deflexion into that curve from its rectilinear path is occasioned by the gravity of the stone towards the earth; that is, by an occult quality. But, now, somebody more cunning than he may come to explain the cause after this manner. He will suppose a certain subtile matter, not discernible by our sight, our touch, or any other of our senses, which fills the spaces which are near and contiguous to the superficies of the earth; and that this matter is carried with different directions, and various, and often contrary motions, describing parabolic curves. Then see how easily he may account for the deflexion of the stone above spoken of. The stone, says he, floats in this subtile fluid, and, following its motion, cannot chuse but describe the same figure. But the fluid moves in parabolic curves, and therefore the stone must move in a parabola of course. Would not the acuteness of this philosopher be thought very extraordinary, who could deduce the appearances of nature from mechanical causes, matter, and motion, so clearly that the meanest man may understand it? Or, indeed, should not we smile to see this new *Galileo* taking so much mathematical pains to introduce occult qualities into philosophy, from whence they have been so happily excluded? But I am ashamed to dwell so long upon trifles.

The sum of the matter is this: the number of the comets is certainly very great; their motions are perfectly regular, and observe the same laws with those of the planets. The orbits in which they move are conic sections, and those very eccentric. They move every way towards all parts of the heavens, and pass through the planetary regions with all possible freedom; and their motion is often contrary to the or-

der of the figns. Thefe phænomena are moft evidently confirmed by aftronomical obfervations, and cannot be accounted for by vortices. Nay, indeed, they are utterly irreconcileable with the vortices of the planets. There can be no room for the motions of the comets, unlefs the celeftial fpaces be entirely cleared of that fictitious matter.

For if the planets are carried about the fun in vortices, the parts of the vortices which immediately furround every planet muft be of the fame denfity with the planet, as was fhewn above; therefore all the matter contiguous to the perimeter of the *magnus orbis* muft be of the fame denfity as the earth. But now that which lies between the *magnus orbis* and the orb of Saturn muft have either an equal or greater denfity; for, to make the conftitution of the vortex permanent, the parts of lefs denfity muft lie near the centre, and thofe of greater denfity muft go farther from it; for fince the periodic times of the planets are in the fefquiplicate ratio of their diftances from the fun, the periods of the parts of the vortices muft alfo preferve the fame ratio. Thence it will follow that the centrifugal forces of the parts of the vortex muft be reciprocally as the fquares of their diftances. Thofe parts, therefore, which are more remote from the centre endeavour to recede from it with lefs force; whence, if their denfity be deficient, they muft yield to the greater force with which the parts that lie nearer the centre endeavour to afcend. Therefore the denfer parts will afcend, and thofe of lefs denfity will defcend; and there will be a mutual change of places, till all the fluid matter in the whole vortex be fo adjufted and difpofed, that, being reduced to an equilibrium, its parts become quiefcent. If two fluids of different denfity be contained in the fame veffel, it will certainly come to pafs that the fluid of greater denfity will fink the loweft; and by a like reafoning it follows that the denfer parts of the vortex by their greater centrifugal force will afcend to the higheft places. Therefore all that far greater part of the vortex which lies without the earth's orb will have a denfity, and by confequence a *vis inertiæ* anfwering to the bulk of the matter, which cannot be lefs than the denfity and *vis inertiæ* of the earth. But from hence will

arise a mighty resistance to the passage of the comets, and such as cannot but be very sensible; not to say enough to put a stop to, and absorb, their motions entirely. But now it appears, from the perfectly regular motion of the comets, that they suffer no resistance that is in the least sensible; and therefore that they meet with no matter of any kind that has any resisting force, or, by consequence, any density or *vis inertiæ;* for the resistance of mediums arises either from the *inertiæ* of the matter of the fluid, or from its want of lubricity. That which arises from the want of lubricity is very small, and is scarcely observable in the fluids commonly known, unless they be very tenacious, like oil and honey. The resistance we find in air, water, quicksilver, and the like fluids that are not tenacious, is almost all of the first kind; and cannot be diminished by a greater degree of subtilty, if the density and *vis inertiæ,* to which this resistance is proportional, remains; as is most evidently demonstrated by our author in his noble theory of resistances, in the second book.

Bodies in going on through a fluid communicate their motion to the ambient fluid by little and little, and by that communication lose their own motion, and by losing it are retarded. Therefore the retardation is proportional to the motion communicated; and the communicated motion, when the velocity of the moving body is given, is as the density of the fluid; and therefore the retardation or resistance will be as the same density of the fluid; nor can it be taken away, unless the fluid coming about to the hinder parts of the body restore the motion lost. Now this cannot be done unless the impression of the fluid on the hinder parts of the body be equal to the impression of the fore parts of the body on the fluid; that is, unless the relative velocity with which the fluid pushes the body behind is equal to the velocity with which the body pushes the fluid; that is, unless the absolute velocity of the recurring fluid be twice as great as the absolute velocity with which the fluid is driven forwards by the body; which is impossible. Therefore the resistance of fluids arising from their *vis inertiæ* can by no means be taken away; so that we must conclude that the celestial fluid has no *vis inertiæ,* because it has

no refifting force; that it has no force to communicate motion with, becaufe it has no *vis inertiæ;* that it has no force to produce any change in one or more bodies, becaufe it has no force wherewith to communicate motion; that it has no manner of efficacy, becaufe it has no faculty wherewith to produce any change of any kind. Therefore, certainly, this hypothefis may be juftly called ridiculous, and unworthy a philofopher; fince it is altogether without foundation, and does not in the leaft ferve to explain the nature of things. Thofe who would have the heavens filled with a fluid matter, but fuppofe it void of any *vis inertiæ,* do, indeed, in words, deny a vacuum, but allow it in fact; for fince a fluid matter of that kind can no ways be diftinguifhed from empty fpace, the difpute is now about the names and not the natures of things. If any are fo fond of matter, that they will by no means admit of a fpace void of body, let us confider where they muft come at laft.

For either they will fay, that this conftitution of a world, every where full, was made fo by the will of God, to this end; that the operations of Nature might be affifted every where by a fubtile æther pervading and filling all things (which cannot be faid, however, fince we have fhewn from the phænomena of the comets that this æther is of no efficacy at all); or they will fay, that it became fo by the fame will of God, for fome unknown end; which ought not to be faid, becaufe, for the fame reafon, a different conftitution may be as well fuppofed; or, laftly, they will not fay that it was caufed by the will of God, but by fome neceffity of its nature. Therefore they will at laft fink into the mire of that infamous herd, who dream that all things are governed by Fate, and not by Providence; and that matter exifts by the neceffity of its nature always and every where, being infinite and eternal. But, fuppofing thefe things, it muft be alfo every where uniform; for variety of forms is entirely inconfiftent with neceffity. It muft be alfo unmoved; for if it be neceffarily moved in any determinate direction with any determinate velocity, it will by a like neceffity be moved in a different direction with a different velocity; but it can never move in different directions with dif-

ferent velocities; therefore it must be unmoved. Without all doubt, this world, so diversified with that variety of forms and motions we find in it, could arise from nothing but the perfectly free will of GOD directing and presiding over all.

From this Fountain it is that those laws, which we call the laws of Nature, have flowed; in which there appear many traces, indeed, of the most wise contrivance, but not the least shadow of necessity. These, therefore, we must not seek from uncertain conjectures, but learn them from observations and experiments. He who thinks to find the true principles of physics and the laws of natural things by the force alone of his own mind, and the internal light of his reason, must either suppose that the world exists by necessity, and by the same necessity follows the laws proposed; or, if the order of Nature was established by the will of GOD, that himself, a miserable reptile, can tell what was fittest to be done. All sound and true philosophy is founded on the appearances of things, which, if they draw us ever so much against our wills to such principles as most clearly manifest to us the most excellent counsel and supreme dominion of the Allwise and Almighty Being, those principles are not therefore to be laid aside, because some men may perhaps dislike them. They may call them, if they please, miracles or occult qualities; but names maliciously given ought not to be a disadvantage to the things themselves; unless they will say, at last, that all philosophy ought to be founded in atheism. Philosophy must not be corrupted in complaisance to these men; for the order of things will not be changed.

Fair and equal judges will therefore give sentence in favour of this most excellent method of philosophy, which is founded on experiments and observations. To this method it is hardly to be said or imagined what light, what splendor, hath accrued from this admirable work of our illustrious author, whose happy and sublime genius, resolving the most difficult problems, and reaching to discoveries of which the mind of man was thought incapable before, is deservedly admired by all those who are somewhat more than superficially versed in these matters. The gates are now set open; and by his means we

may freely enter into the knowledge of the hidden ſecrets and wonders of natural things. He has ſo clearly laid open and ſet before our eyes the moſt beautiful frame of the Syſtem of the World, that, if King *Alphonſus* were now alive, he would not complain for want of the graces either of ſimplicity or of harmony in it. Therefore we may now more nearly behold the beauties of Nature, and entertain ourſelves with the delightful contemplation; and, which is the beſt and moſt valuable fruit of philoſophy, be thence incited the more profoundly to reverence and adore the great Maker and Lord of all. He muſt be blind, who, from the moſt wiſe and excellent contrivances of things, cannot ſee the infinite wiſdom and goodneſs of their Almighty Creator; and he muſt be mad and ſenſeleſs who refuſes to acknowledge them.

LIFE

OF

SIR ISAAC NEWTON.

SIR ISAAC NEWTON, one of the greateſt philoſophers and mathematicians the world has produced, was born at Woolſtrop, in Lincolnſhire, on Chriſtmas day, 1642. He was deſcended from the eldeſt branch of the family of Sir John Newton, bart., who were lords of the manor of Woolſtrop, and had been poſſeſſed of the eſtate for about two centuries before, to which they had removed from Weſtley, in the ſame county, but originally they came from the town of Newton, in Lancaſhire. Other accounts ſay, I think more truly, that he was the only child of Mr. John Newton, of Coleſworth, near Grantham, in Lincolnſhire, who had there an eſtate of about 120l. a year, which he kept in his own hands. His mother was of the antient and opulent family of the Ayſcoughs, or Aſkews, of the ſame county. Our author loſing his father while he was very young, the care of his education devolved on his mother, who, though ſhe married again after his father's death, did not neglect to improve by a liberal education the promiſing genius that was obſerved in her ſon. At 12 years of age, by the advice of his maternal uncle, he was ſent to the grammar ſchool at Grantham, where he made a good proficiency in the languages, and laid the foundation of his future ſtudies. Even here was obſerved in him a ſtrong inclination to figures and philoſophical ſubjects. One trait of this early diſpoſition is told of him: he had then a rude method of meaſuring the force of the wind blowing againſt him, by obſerving how much farther he could leap in the direction of the wind, or blowing on his back, than he could leap the contrary way, or oppoſed to the wind: an early mark of his original infantine genius.

After a few years ſpent here, his mother took him home intending, as ſhe had no other child, to have the pleaſure of

his company; and that, after the manner of his father before him, he ſhould occupy his own eſtate.

But, inſtead of minding the markets, or the buſineſs of the farm, he was always ſtudying and poring over his books, even by ſtealth, from his mother's knowledge. On one of theſe occaſions his uncle diſcovered him one day in a hay-loft at Grantham, whither he had been ſent to the market, working a mathematical problem; and having otherwiſe obſerved the boy's mind to be uncommonly bent upon learning, he prevailed upon his ſiſter to part with him; and he was accordingly ſent, in 1660, to Trinity College, in Cambridge, where his uncle, having himſelf been a member of it, had ſtill many friends. Iſaac was ſoon taken notice of by Dr. Barrow, who was ſoon after appointed the firſt Lucaſian profeſſor of mathematics; and, obſerving his bright genius, contracted a great friendſhip for him. At his outſetting here, Euclid was firſt put into his hands, as uſual, but that author was ſoon diſmiſſed; ſeeming to him too plain and eaſy, and unworthy of taking up his time. He underſtood him almoſt before he read him; and a caſt of his eye upon the contents of his theorems was ſufficient to make him maſter of them: and as the analytical method of Deſcartes was then much in vogue, he particularly applied to it, and Kepler's Optics, &c. making ſeveral improvements on them, which he entered upon the margins of the books as he went on, as his cuſtom was in ſtudying any author.

Thus he was employed till the year 1664, when he opened a way into his new method of Fluxions and Infinite Series; and the ſame year took the degree of bachelor of arts. In the mean time, obſerving that the mathematicians were much engaged in the buſineſs of improving teleſcopes, by grinding glaſſes into one of the figures made by the three ſections of a cone, upon the principle then generally entertained that light was homogeneous, he ſet himſelf to grinding of optic glaſſes, of other figures than ſpherical, having as yet no diſtruſt of the homogeneous nature of light: but, not hitting preſently upon any thing in this attempt to ſatisfy his mind, he procured a glaſs priſm, that he might try the celebrated

phenomena of colours, discovered by Grimaldi not long before. He was much pleased at first with the vivid brightness of the colours produced by this experiment; but, after a while, considering them in a philosophical way, with that circumspection which was natural to him, he was surprised to see them in an oblong form, which, according to the received rule of refractions, ought to be circular. At first he thought the irregularity might possibly be no more than accidental; but this was what he could not leave without farther enquiry: accordingly, he soon invented an infallible method of deciding the question, and the result was, his *New Theory of Light and Colours*.

However, the theory alone, unexpected and surprising as it was, did not satisfy him; he rather considered the proper use that might be made of it for improving telescopes, which was his first design. To this end, having now discovered that light was not homogeneous, but an heterogeneous mixture of differently refrangible rays, he computed the errors arising from this different refrangibility; and, finding them to exceed some hundreds of times those occasioned by the circular figure of the glasses, he threw aside his glass works, and took reflections into consideration. He was now sensible that optical instruments might be brought to any degree of perfection desired, in case there could be found a reflecting substance which would polish as finely as glass, and reflect as much light as glass transmits, and the art of giving it a parabolical figure he also attained: but these seemed to him very great difficulties; nay, he almost thought them insuperable, when he farther considered, that every irregularity in a reflecting superficies makes the rays stray five or six times more from their due course, than the like irregularities in a refracting one.

Amidst these speculations, he was forced from Cambridge, in 1665, by the plague; and it was more than two years before he made any farther progress in the subject. However, he was far from passing his time idly in the country; on the contrary, it was here, at this time, that he first started the hint that gave rise to the System of the World, which is the

main subject of the *Principia*. In his retirement, he was sitting alone in a garden, when some apples falling from a tree, led his thoughts upon the subject of gravity; and, reflecting on the power of that principle, he began to consider, that as this power is not found to be sensibly diminished at the remotest distance from the centre of the earth to which we can rise, neither at the tops of the loftiest buildings, nor on the summits of the highest mountains, it appeared to him reasonable to conclude that this power must extend much farther than is usually thought. "Why not as high as the moon?" said he to himself; "and if so, her motion must be influenced by it; perhaps she is retained in her orbit by it: however, though the power of gravity is not sensibly weakened in the little change of distance at which we can place ourselves from the centre of the earth, yet it is very possible that, at the height of the moon, this power may differ in strength much from what it is here." To make an estimate what might be the degree of this diminution, he considered with himself, that, if the moon be retained in her orbit by the force of gravity, no doubt the primary planets are carried about the sun by the like power; and, by comparing the periods of the several planets with their distances from the sun, he found, that, if any power like gravity held them in their courses, its strength must decrease in the duplicate proportion of the increase of distance. This he concluded by supposing them to move in perfect circles, concentric to the sun, from which the orbits of the greatest part of them do not much differ. Supposing, therefore, the force of gravity, when extended to the moon, to decrease in the same manner, he computed whether that force would be sufficient to keep the moon in her orbit.

In this computation, being absent from books, he took the common estimate in use among the geographers and our seamen, before Norwood had measured the earth, namely, that 60 miles make one degree of latitude; but as that is a very erroneous supposition, each degree containing about 69½ of our English miles, his computation upon it did not make the power of gravity, decreasing in a duplicate proportion to

the diſtance, anſwerable to the power which retained the moon in her orbit: whence he concluded that ſome other cauſe muſt at leaſt join with the action of the power of gravity on the moon. For this reaſon he laid aſide, for that time, any farther thoughts upon the matter. Mr. Whiſton (in his Memoirs, p. 33) ſays, he told him that he thought Deſcartes's vortices might concur with the action of gravity.

Nor did he reſume this enquiry on his return to Cambridge, which was ſhortly after. The truth is, his thoughts were now engaged upon his newly projected reflecting teleſcope, of which he made a ſmall ſpecimen, with a metallic reflector ſpherically concave. It was but a rude eſſay, chiefly defective by the want of a good poliſh for the metal. This inſtrument is now in the poſſeſſion of the Royal Society. In 1667 he was choſen fellow of his college, and took the degree of maſter of arts. And in 1669 Dr. Barrow reſigned to him the mathematical chair at Cambridge, the buſineſs of which appointment interrupted for a while his attention to the teleſcope: however, as his thoughts had been for ſome time chiefly employed upon optics, he made his diſcoveries in that ſcience the ſubject of his lectures, for the firſt three years after he was appointed Mathematical Profeſſor: and having now brought his *Theory of Light and Colours* to a conſiderable degree of perfection, and having been elected a Fellow of the Royal Society in Jan. 1672, he communicated it to that body, to have their judgment upon it; and it was afterwards publiſhed in their Tranſactions, viz. of Feb. 19, 1672. This publication occaſioned a diſpute upon the truth of it, which gave him ſo much uneaſineſs, that he reſolved not to publiſh any thing farther for a while upon the ſubject; and in that reſolution he laid up his *Optical Lectures*, although he had prepared them for the preſs. And the *Analyſis by Infinite Series*, which he had intended to ſubjoin to them, unhappily for the world, underwent the ſame fate, and for the ſame reaſon.

In this temper he reſumed his teleſcope; and obſerving that there was no abſolute neceſſity for the parabolic figure of the glaſſes, ſince, if metals could be ground truly ſpherical, they

would be able to bear as great apertures as men could give a polish to, he completed another instrument of the same kind. This answering the purpose so well, as, though only half a foot in length, to shew the planet Jupiter distinctly round, with his four satellites, and also Venus horned, he sent it to the Royal Society, at their request, together with a description of it, with farther particulars; which were published in the Philosophical Transactions for March 1672. Several attempts were also made by that society to bring it to perfection; but, for want of a proper composition of metal, and a good polish, nothing succeeded, and the invention lay dormant, till Hadley made his Newtonian telescope in 1723. At the request of Leibnitz, in 1676, he explained his invention of Infinite Series, and took notice how far he had improved it by his Method of Fluxions, which however he still concealed, and particularly on this occasion, by a transposition of the letters that make up the two fundamental propositions of it into an alphabetical order; the letters concerning which are inserted in Collins's Commercium Epistolicum, printed 1712. In the winter between the years 1676 and 1677 he found out the grand proposition, that, by a centripetal force acting reciprocally as the square of the distance, a planet must revolve in an ellipsis, about the centre of force placed in its lower focus, and, by a radius drawn to that centre, describe areas proportional to the times. In 1680 he made several astronomical observations upon the comet that then appeared; which, for some considerable time, he took not to be one and the same, but two different comets; and upon this occasion several letters passed between him and Mr. Flamsted.

He was still under this mistake, when he received a letter from Dr. Hook, explaining the nature of the line described by a falling body, supposed to be moved circularly by the diurnal motion of the earth, and perpendicularly by the power of gravity. This letter put him upon enquiring anew what was the real figure in which such a body moved; and that enquiry, convincing him of another mistake which he had before fallen into concerning that figure, put him upon resuming his former thoughts with regard to the moon; and Picart

having not long before, viz. in 1679, meaſured a degree of the earth with ſufficient accuracy, by uſing his meaſures, that planet appeared to be retained in her orbit by the ſole power of gravity; and conſequently that this power decreaſes in the duplicate ratio of the diſtance; as he had formerly conjectured. Upon this principle he found the line deſcribed by a falling body to be an ellipſis, having one focus in the centre of the earth. And finding by this means that the primary planets really moved in ſuch orbits as Kepler had ſuppoſed, he had the ſatisfaction to ſee that this enquiry, which he had undertaken at firſt out of mere curioſity, could be applied to the greateſt purpoſes. Hereupon he drew up about a dozen propoſitions relating to the motion of the primary planets round the ſun, which were communicated to the Royal Society in the latter end of 1683. This coming to be known to Dr. Halley, that gentleman, who had attempted the demonſtration in vain, applied, in Auguſt 1684, to Newton, who aſſured him that he had abſolutely completed the proof. This was alſo regiſtered in the books of the Royal Society; at whoſe earneſt ſolicitation Newton finiſhed the work, which was printed under the care of Dr. Halley, and came out about Midſummer 1687, under the title of, *Philoſophiæ Naturalis Principia Mathematica,* containing, in the third book, the Cometic Aſtronomy, which had been lately diſcovered by him, and now made its firſt appearance in the world: a work which may be looked upon as the production of a celeſtial intelligence rather than of a man.

This work, however, in which the great author has built a new ſyſtem of natural philoſophy upon the moſt ſublime geometry, did not meet at firſt with all the applauſe it deſerved, and was one day to receive. Two reaſons concurred in producing this effect: Deſcartes had then got full poſſeſſion of the world. His philoſophy was, indeed, the creature of a fine imagination, gaily dreſſed out: he had given her likewiſe ſome of Nature's fine features, and painted the reſt to a ſeeming likeneſs of her. On the other hand, Newton had with an unparalleled penetration, and force of genius, purſued Nature up to her moſt ſecret abode, and was intent to demonſtrate

her residence to others, rather than anxious to describe particularly the way by which he arrived at it himself: he finished his piece in that elegant conciseness which had justly gained the antients an universal esteem. In fact, the consequences flow with such rapidity from the Principles, that the reader is often left to supply a long chain of reasoning to connect them; so that it required some time before the world could understand it. The best mathematicians were obliged to study it with care, before they could make themselves masters of it; and those of a lower rank durst not venture upon it, till encouraged by the testimonies of the more learned. But, at last, when its value came to be sufficiently known, the approbation which had been so slowly gained became universal, and nothing was to be heard from all quarters but one general burst of admiration. "Does Mr. Newton eat, drink, or sleep, like other men?" says the marquis de l'Hospital, one of the greatest mathematicians of the age, to the English who visited him. "I represent him to myself as a celestial genius entirely disengaged from matter."

In the midst of these profound mathematical researches, just before his Principia went to the press, in 1686, the privileges of the university being attacked by James the 2d, Newton appeared among its most strenuous defenders, and was on that occasion appointed one of their delegates to the high-commission court; and they made such a defence, that James thought proper to drop the affair. Our author was also chosen one of their members for the Convention-Parliament in 1688, in which he sat till it was dissolved.

Newton's merit was well known to Mr. Montague, then chancellor of the exchequer, and afterwards earl of Halifax, who had been bred at the same college with him; and, when he undertook the great work of recoining the money, he fixed his eye upon Newton for an assistant in it; and accordingly, in 1696, he was appointed warden of the mint, in which employment he rendered very signal service to the nation. And three years after he was promoted to be master of the mint, a place worth 12 or 15 hundred pounds per annum, which he held till his death. Upon this promotion, he appointed Mr.

Whiston his deputy in the mathematical professorship at Cambridge, giving him the full profits of the place, which appointment itself he also procured for him in 1703. The same year our author was chosen president of the Royal Society, in which chair he sat for 25 years, namely, till the time of his death; and he had been chosen a member of the Royal Academy of Sciences at Paris in 1699, as soon as the new regulation was made for admitting foreigners into that society.

Ever since the first discovery of the heterogeneous mixture of light, and the production of colours thence arising, he had employed a good part of his time in bringing the experiment, upon which the theory is founded, to a degree of exactness that might satisfy himself. The truth is, this seems to have been his favourite invention; 30 years he had spent in this arduous task before he published it in 1704. In infinite series and fluxions, and in the power and rule of gravity in preserving the solar system, there had been some though distant hints given by others before him: whereas in dissecting a ray of light into its primary constituent particles, which then admitted of no farther separation; in the discovery of the different refrangibility of these particles thus separated; and that these constituent rays had each its own peculiar colour inherent in it; that rays falling in the same angle of incidence have alternate fits of reflection and refraction; that bodies are rendered transparent by the minuteness of their pores, and become opaque by having them large; and that the most transparent body, by having a great thinness, will become less pervious to the light; in all these, which make up his new theory of light and colours, he was absolutely and entirely the first starter; and as the subject is of the most subtle and delicate nature, he thought it necessary to be himself the last finisher of it.

In fact, the affair that chiefly employed his researches for so many years was far from being confined to the subject of light alone. On the contrary, all that we know of natural bodies seemed to be comprehended in it: he had found out that there was a natural action at a distance between light and other bodies, by which both the reflections and refractions,

as well as inflections, of the former, were constantly produced. To ascertain the force and extent of this principle of action, was what had all along engaged his thoughts, and what after all, by its extreme subtlety, escaped his most penetrating spirit. However, though he has not made so full a discovery of this principle, which directs the course of light, as he has in regard to the power by which the planets are kept in their courses, yet he gave the best directions possible for such as should be disposed to carry on the work, and furnished matter abundantly sufficient to animate them to the pursuit. He has indeed hereby opened a way of passing from optics to an entire system of physics; and, if we look upon his queries as containing the history of a great man's first thoughts, even in that view they must be always at least entertaining and curious.

This same year, and in the same book with his Optics, he published, for the first time, his Method of Fluxions. It has been already observed, that these two inventions were intended for the public so long before as 1672; but were laid by then, in order to prevent his being engaged on that account in a dispute about them. And it is not a little remarkable, that even now this last piece proved the occasion of another dispute, which continued for many years. Ever since 1684, Leibnitz had been artfully working the world into an opinion, that he first invented this method.—Newton saw his design from the beginning, and had sufficiently obviated it in the first edition of the Principia, in 1687 (viz. in the Scholium to the 2d lemma of the 2d book); and with the same view, when he now published that method, he took occasion to acquaint the world, that he invented it, in the years 1665 and 1666. In the Acta Eruditorum of Leipsic, where an account is given of this book, the author of that account ascribed the invention to Leibnitz, intimating that Newton borrowed it from him. Dr. Keill, the astronomical professor at Oxford, undertook Newton's defence; and after several answers on both sides, Leibnitz complaining to the Royal Society, this body appointed a committee of their members to examine the merits of the case. These, after

considering all the papers and letters relating to the point in controversy, decided in favour of Newton and Keill; as is related at large in the life of this last mentioned gentleman; and these papers themselves were published in 1712, under the title of Commercium Epistolicum Johannis Collins, 8vo.

In 1705, the honour of knighthood was conferred upon our author by queen Anne, in consideration of his great merit. And in 1714 he was applied to by the House of Commons for his opinion upon a new method of discovering the longitude at sea by signals, which had been laid before them by Ditton and Whiston, in order to procure their encouragement; but the petition was thrown aside upon reading Newton's paper delivered to the committee.

The following year, 1715, Leibnitz, with the view of bringing the world more easily into the belief that Newton had taken the method of fluxions from his Differential method, attempted to foil his mathematical skill by the famous problem of the trajectories, which he therefore proposed to the English by way of challenge; but the solution of this, though the most difficult proposition he was able to devise, and what might pass for an arduous affair to any other, yet was hardly any more than an amusement to Newton's penetrating genius: he received the problem at four o'clock in the afternoon, as he was returning from the Mint; and, though extremely fatigued with business, yet he finished the solution before he went to bed.

As Leibnitz was privy-counsellor of justice to the elector of Hanover, so when that prince was raised to the British throne, Newton came more under the notice of the court; and it was for the immediate satisfaction of George the First, that he was prevailed on to put the last hand to the dispute about the invention of Fluxions. In this court, Caroline princess of Wales, afterwards queen consort to George the Second, happened to have a curiosity for philosophical enquiries; no sooner, therefore, was she informed of our author's attachment to the house of Hanover, than she engaged his conversation, which soon endeared him to her. Here she found in every difficulty that full satisfaction which she had in vain sought

for elsewhere; and she was often heard to declare publicly, that she thought herself happy in coming into the world at a juncture of time which put it in her power to converse with him. It was at this princess's solicitation, that he drew up an abstract of his Chronology; a copy of which was at her request communicated, about 1718, to signior Conti, a Venetian nobleman, then in England, upon a promise to keep it secret. But notwithstanding this promise, the abbé, who while here had also affected to shew a particular friendship for Newton, though privately betraying him as much as lay in his power to Leibnitz, was no sooner got across the water into France, than he dispersed copies of it, and procured an antiquary to translate it into French, as well as to write a confutation of it. This, being printed at Paris in 1725, was delivered as a present from the bookseller that printed it to our author, that he might obtain, as was said, his consent to the publication; but though he expressly refused such consent, yet the whole was published the same year. Hereupon Newton found it necessary to publish a Defence of himself, which was inserted in the Philosophical Transactions. Thus he, who had so much all his life long been studious to avoid disputes, was unavoidably all his life time, in a manner, involved in them; nor did this last dispute even finish at his death, which happened the year following. Newton's paper was republished in 1726 at Paris, in French, with a letter of the abbé Conti in answer to it; and the same year some dissertations were printed there by father Souciet against Newton's Chronological Index, an answer to which was inserted by Halley in the Philosophical Transactions, numb. 397.

Some time before this business, in his 80th year, our author was seized with an incontinence of urine, thought to proceed from the stone in the bladder, and deemed to be incurable. However, by the help of a strict regimen and other precautions, which till then he never had occasion for, he procured considerable intervals of ease during the five remaining years of his life. Yet he was not free from some severe paroxysms, which even forced out large drops of sweat that ran down his face. In these circumstances he was never

obſerved to utter the leaſt complaint, nor expreſs the leaſt impatience; and as ſoon as he had a moment's eaſe, he would ſmile and talk with his uſual cheerfulneſs. He was now obliged to rely upon Mr. Conduit, who had married his niece, for the diſcharge of his office in the Mint. Saturday morning, March 18, 1727, he read the newſpapers, and diſcourſed a long time with Dr. Mead, his phyſician, having then the perfect uſe of all his ſenſes and his underſtanding; but that night he entirely loſt them all, and, not recovering them afterwards, died the Monday following, March 20, in the 85th year of his age. His corpſe lay in ſtate in the Jeruſalem-chamber, and on the 28th was conveyed into Weſtminſter-abbey, the pall being ſupported by the lord chancellor, the dukes of Montroſe and Roxburgh, and the earls of Pembroke, Suſſex, and Macclesfield. He was interred near the entrance into the choir on the left hand, where a ſtately monument is erected to his memory, with a moſt elegant inſcription upon it.

Newton's character has been attempted by M. Fontenelle and Dr. Pemberton, the ſubſtance of which is as follows — He was of a middle ſtature, and ſomewhat inclined to be fat in the latter part of his life. His countenance was pleaſing and venerable at the ſame time; eſpecially when he took off his peruke, and ſhewed his white hair, which was pretty thick. He never made uſe of ſpectacles, and loſt but one tooth during his whole life. Biſhop Atterbury ſays, that, in the whole air of Sir Iſaac's face and make, there was nothing of that penetrating ſagacity which appears in his compoſitions; that he had ſomething rather languid in his look and manner, which did not raiſe any great expectation in thoſe who did not know him.

His temper, it is ſaid, was ſo equal and mild, that no accident could diſturb it. A remarkable inſtance of which is related as follows. Sir Iſaac had a favourite little dog, which he called Diamond. Being one day called out of his ſtudy into the next room, Diamond was left behind. When Sir Iſaac returned, having been abſent but a few minutes, he had the mortification to find, that Diamond, having overſet a

lighted candle among ſome papers, the nearly finiſhed labour of many years was in flames, and almoſt conſumed to aſhes. This loſs, as Sir Iſaac was then very far advanced in years, was irretrievable; yet, without once ſtriking the dog, he only rebuked him with this exclamation, "Oh, Diamond! Diamond! thou little knoweſt the miſchief thou haſt done!"

He was indeed of ſo meek and gentle a diſpoſition, and ſo great a lover of peace, that he would rather have choſen to remain in obſcurity, than to have the calm of life ruffled by thoſe ſtorms and diſputes which genius and learning always draw upon thoſe that are the moſt eminent for them.

From his love of peace, no doubt, aroſe that unuſual kind of horror which he felt for all diſputes: a ſteady unbroken attention, free from thoſe frequent recoilings inſeparably incident to others, was his peculiar felicity; he knew it, and he knew the value of it. No wonder, then, that controverſy was looked on as his bane. When ſome objections, haſtily made to his diſcoveries concerning light and colours, induced him to lay aſide the deſign he had taken of publiſhing his Optical Lectures, we find him reflecting on that diſpute, into which he had been unavoidably drawn, in theſe terms: "I blamed my own imprudence for parting with ſo real a bleſſing as my quiet, to run after a ſhadow." It is true this ſhadow, as Fontenelle obſerves, did not eſcape him afterwards, nor did it coſt him that quiet which he ſo much valued, but proved as much a real happineſs to him as his quiet itſelf; yet this was a happineſs of his own making: he took a reſolution, from theſe diſputes, not to publiſh any more concerning that theory till he had put it above the reach of controverſy, by the exacteſt experiments, and the ſtricteſt demonſtrations; and accordingly it has never been called in queſtion ſince. In the ſame temper, after he had ſent the manuſcript to the Royal Society, with his conſent to the printing of it by them, yet upon Hook's injuriouſly inſiſting that he himſelf had demonſtrated Kepler's problem before our author, he determined, rather than be involved again in a controverſy, to ſuppreſs the third book; and he was very hardly prevailed upon to alter that reſolution. It is true, the public was

thereby a gainer; that book, which is indeed no more than a corollary of ſome propoſitions in the firſt, being originally drawn up in the popular way, with a deſign to publiſh it in that form; whereas he was now convinced that it would be beſt not to let it go abroad without a ſtrict demonſtration.

In contemplating his genius, it preſently becomes a doubt, which of theſe endowments had the greateſt ſhare,—ſagacity, penetration, ſtrength, or diligence; and, after all, the mark that ſeems moſt to diſtinguiſh it is, that he himſelf made the juſteſt eſtimation of it, declaring, that if he had done the world any ſervice, it was due to nothing but induſtry and patient thought; that he kept the ſubject of conſideration conſtantly before him, and waited till the firſt dawning opened gradually, by little and little, into a full and clear light. It is ſaid, that, when he had any mathematical problems or ſolutions in his mind, he would never quit the ſubject on any account. And his ſervant has ſaid, when he has been getting up in a morning, he has ſometimes begun to dreſs, and with one leg in his breeches, ſat down again on the bed, where he has remained for hours before he has got his clothes on: and that dinner has been often three hours ready for him before he could be brought to table. Upon this head ſeveral little anecdotes are related; among which is the following: Doctor Stukely coming in accidentally, one day, when Newton's dinner was left for him upon the table, covered up, as uſual, to keep it warm till he could find it convenient to come to table; the doctor, lifting the cover, found under it a chicken, which he preſently ate, putting the bones in the diſh, and replacing the cover. Some time after Newton came into the room, and after the uſual compliments ſat down to his dinner; but on taking up the cover, and ſeeing only the bones of the fowl left, he obſerved, with ſome little ſurpriſe, "I thought I had not dined, but I now find that I have."

After all, notwithſtanding his anxious care to avoid every occaſion of breaking his intenſe application to ſtudy, he was at a great diſtance from being ſteeped in philoſophy. On the contrary, he could lay aſide his thoughts, though engaged in

the moſt intricate reſearches, when his other affairs required his attention; and, as ſoon as he had leiſure, reſume the ſubject at the point where he had left off. This he ſeems to have done not ſo much by any extraordinary ſtrength of memory, as by the force of his inventive faculty, to which every thing opened itſelf again with eaſe, if nothing intervened to ruffle him. The readineſs of his invention made him not think of putting his memory much to the trial; but this was the offspring of a vigorous intenſeneſs of thought, out of which he was but a common man. He ſpent, therefore, the prime of his age in thoſe abſtruſe reſearches, when his ſituation in a college gave him leiſure, and while ſtudy was his proper buſineſs. But as ſoon as he was removed to the mint, he applied himſelf chiefly to the duties of that office; and ſo far quitted mathematics and philoſophy as not to engage in any purſuits of either kind afterwards.

Dr. Pemberton obſerves, that though his memory was much decayed in the laſt years of his life, yet he perfectly underſtood his own writings; contrary to what I had formerly heard, ſays the doctor, in diſcourſe from many perſons. This opinion of their's might ariſe perhaps from his not being always ready at ſpeaking on theſe ſubjects, when it might be expected he ſhould. But on this head it may be obſerved, that great geniuſes are often liable to be abſent, not only in relation to common life, but with regard to ſome of the parts of ſcience that they are beſt informed of: inventors ſeem to treaſure up in their minds what they have found out after another manner than thoſe do the ſame things, who have not this inventive faculty. The former, when they have occaſion to produce their knowledge, are in ſome meaſure obliged immediately to inveſtigate part of what they want; and for this they are not equally fit at all times; from whence it has often happened, that ſuch as retain things chiefly by means of a very ſtrong memory, have appeared off-hand more expert than the diſcoverers themſelves.

It was evidently owing to the ſame inventive faculty that Newton, as this writer found, had read fewer of the modern mathematicians than one could have expected; his own pro-

digious invention readily supplying him with what he might have occasion for in the pursuit of any subject he undertook. However, he often censured the handling of geometrical subjects by algebraic calculations; and his book of algebra he called by the name of *Universal Arithmetic*, in opposition to the injudicious title of *Geometry* which Descartes had given to the treatise in which he shews how the geometrician may assist his invention by such kind of computations. He frequently praised Slusius, Barrow, and Huygens, for not being influenced by the false taste which then began to prevail. He used to commend the laudable attempt of Hugo d'Omerique to restore the antient analysis; and very much esteemed Apollonius's book *De Sectione Rationis*, for giving us a clearer notion of that analysis than we had before. Dr. Barrow may be esteemed as having shewn a compass of invention equal, if not superior, to any of the moderns, our author only excepted; but Newton particularly recommended Huygens's style and manner; he thought him the most elegant of any mathematical writer of modern times, and the truest imitator of the antients. Of their taste and mode of demonstration our author always professed himself a great admirer; and even censured himself for not following them yet more closely than he did; and spoke with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Descartes, and other algebraic writers, before he had considered the Elements of Euclid with that attention which so excellent a writer deserves.

But if this was a fault, it is certain it was a fault to which we owe both his great inventions in speculative mathematics, and the doctrine of Fluxions and Infinite Series. And perhaps this might be one reason why his particular reverence for the antients is omitted by Fontenelle, who, however, certainly makes some amends by that just eulogium which he makes of our author's modesty, which amiable quality he represents as standing foremost in the character of this great man's mind and manners. It was in reality greater than can be easily imagined, or will be readily believed: yet it always continued so without any alteration; though the whole world,

says Fontenelle, conspired against it; let us add, though he was thereby robbed of his invention of Fluxions.' Nicholas Mercator publishing his *Logarithmotechnia* in 1668, where he gave the quadrature of the hyperbola by an infinite series, which was the first appearance in the learned world of a series of this sort drawn from the particular nature of the curve, and that in a manner very new and abstracted; Dr. Barrow, then at Cambridge, where Mr. Newton, then about 26 years of age, resided, recollected, that he had met with the same thing in the writings of that young gentleman; and there not confined to the hyperbola only, but extended, by general forms, to all sorts of curves, even such as are mechanical; to their quadratures, their rectifications, and their centres of gravity; to the solids formed by their rotations, and to the superficies of those solids; so that, when their determinations were possible, the series stopped at a certain point, or at least their sums were given by stated rules; and if the absolute determinations were impossible, they could yet be infinitely approximated; which is the happiest and most refined method, says Fontenelle, of supplying the defects of human knowledge that man's imagination could possibly invent. To be master of so fruitful and general a theory was a mine of gold to a geometrician; but it was a greater glory to have been the discoverer of so surprising and ingenious a system. So that Newton, finding by Mercator's book that he was in the way to it, and that others might follow in his track, should naturally have been forward to open his treasures, and secure the property, which consisted in making the discovery; but he contented himself with his treasure which he had found, without regarding the glory. What an idea does it give us of his unparalleled modesty, when we find him declaring, that he thought Mercator had entirely discovered his secret, or that others would, before he should become of a proper age for writing! His manuscript upon Infinite Series was communicated to none but Mr. John Collins and the lord Brounker, then President of the Royal Society, who had also done something in this way himself; and even that had not

been complied with, but for Dr. Barrow, who would not ſuffer him to indulge his modeſty ſo much as he deſired.

It is farther obſerved, concerning this part of his character, that he never talked either of himſelf or others, nor ever behaved in ſuch a manner as to give the moſt malicious cenſurers the leaſt occaſion even to ſuſpect him of vanity. He was candid and affable, and always put himſelf upon a level with his company. He never thought either his merit or his reputation ſufficient to excuſe him from any of the common offices of ſocial life. No ſingularities, either natural or affected, diſtinguiſhed him from other men. Though he was firmly attached to the church of England, he was averſe to the perſecution of the non-conformiſts. He judged of men by their manners; and the true ſchiſmatics, in his opinion, were the vicious and the wicked. Not that he confined his principles to natural religion, for it is ſaid he was thoroughly perſuaded of the truth of Revelation; and amidſt the great variety of books which he had conſtantly before him, that which he ſtudied with the greateſt application was the Bible, at leaſt in the latter years of his life; and he underſtood the nature and force of moral certainty as well as he did that of a ſtrict demonſtration.

Sir Iſaac did not neglect the opportunities of doing good, when the revenues of his patrimony and a profitable employment, improved by a prudent œconomy, put it in his power. We have two remarkable inſtances of his bounty and generoſity; one to Mr. Maclaurin, extra profeſſor of mathematics at Edinburgh, to encourage whoſe appointment he offered 20 pounds a year to that office; and the other to his niece Barton, upon whom he had ſettled an annuity of 100 pounds per annum. When decency upon any occaſion required expence and ſhew, he was magnificent without grudging it, and with a very good grace; at all other times, that pomp which ſeems great to low minds only, was utterly retrenched, and the expence reſerved for better uſes.

Newton never married; and it has been ſaid, that "perhaps he never had leiſure to think of it; that, being immerſed in profound ſtudies during the prime of his age, and

afterwards engaged in an employment of great importance, and even quite taken up with the company which his merit drew to him, he was not fenfible of any vacancy in life, nor of the want of a companion at home." Thefe, however, do not appear to be any fufficient reafons for his never marrying, if he had had an inclination fo to do. It is much more likely that he had a conftitutional indifference to the ftate, and even to the fex in general; and it has even been faid of him, that he never once knew woman.—He left at his death, it feems, 32 thoufand pounds; but he made no will; which, Fontenelle tells us, was becaufe he thought a legacy was no gift. As to his works, befides what were publifhed in his life-time, there were found after his death, among his papers, feveral difcourfes upon the fubjects of Antiquity, Hiftory, Divinity, Chemiftry, and Mathematics; feveral of which were publifhed at different times, as appears from the following catalogue of all his works; where they are ranked in the order of time in which thofe upon the fame fubject were publifhed.

1. Several papers relating to his *Telefcope,* and his *Theory of Light and Colours,* printed in the Philofophical Tranfactions, numbs. 80, 81, 82, 83, 84, 85, 88, 96, 97, 110, 121, 123, 128; or vols. 6, 7, 8, 9, 10, 11.

2. *Optics, or a Treatife of the Reflections, Refractions, and Inflections, and the Colours of Light;* 1704, 4to.—A Latin tranflation by Dr. Clarke; 1706, 4to.—And a French tranflation by Pet. Cofte, Amft. 1729, 2 vols. 12mo.—Befide feveral Englifh editions in 8vo.

3. *Optical Lectures;* 1728, 8vo. Alfo in feveral Letters to Mr. Oldenburg, fecretary of the Royal Society, inferted in the General Dictionary, under our author's article.

4. *Lectiones Opticæ;* 1729, 4to.

5. *Naturalis Philofophiæ Principia Mathematica;* 1687, 4to.—A fecond edition in 1713, with a Preface, by Roger Cotes.—The third edition in 1726, under the direction of Dr. Pemberton.—An Englifh tranflation, by Motte, 1729, 2 vols. 8vo. printed in feveral editions of his works, in different nations, particularly an edition, with a large Com-

mentary, by the two learned Jesuits, Le Seur and Jacquier, in 4 vols. 4to, in 1739, 1740, and 1742.

6. *A System of the World,* translated from the Latin original; 1727, 8vo.—This, as has been already observed, was at first intended to make the third book of his Principia.—An English translation by Motte, 1729, 8vo.

7. *Several Letters* to Mr. Flamsted, Dr. Halley, and Mr. Oldenburg.—See our author's article in the General Dictionary.

8. *A Paper concerning the Longitude;* drawn up by order of the House of Commons; ibid.

9. *Abregé de Chronologie,* &c; 1726, under the direction of the abbé Conti, together with some observations upon it.

10. *Remarks upon the Observations made upon a Chronological Index of Sir I. Newton, &c.* Philos. Trans. vol. 33.—See also the same, vol. 34 and 35, by Dr. Halley.

11. *The Chronology of Antient Kingdoms amended, &c.*; 1728, 4to.

12. *Arithmetica Universalis, &c.*; under the inspection of Mr. Whiston, Cantab. 1707, 8vo. Printed, I think, without the author's consent, and even against his will: an offence which it seems was never forgiven. There are also English editions of the same, particularly one by Wilder, with a Commentary, in 1769, 2 vols. 8vo. And a Latin edition, with a Commentary, by Castilion, 2 vols. 4to, Amst. &c.

13. *Analysis per Quantitatum Series, Fluxiones, et Differentias, cum Enumeratione Linearum Tertii Ordinis;* 1711, 4to; under the inspection of W. Jones, Esq. F. R. S.—The last tract had been published before, together with another on the *Quadrature of Curves,* by the Method of Fluxions, under the title of *Tractatus duo de Speciebus & Magnitudine Figurarum Curvilinearum;* subjoined to the first edition of his Optics in 1704; and other letters in the Appendix to Dr. Gregory's Catoptrics, &c. 1735, 8vo.—Under this head may be ranked *Newtoni Genesis Curvarum per Umbras;* Leyden, 1740.

14. *Several Letters relating to his Dispute with Leibnitz*, upon his Right to the Invention of Fluxions; printed in the *Commercium Epistolicum D. Johannis Collins & aliorum de Analysi Promota, jussu Societatis Regiæ editum*; 1712, 8vo.

15. Postscript and Letter of M. Leibnitz to the abbé Conti, with Remarks, and a Letter of his own to that abbé; 1717, 8vo.—To which was added, Raphson's History of Fluxions, as a Supplement.

16. *The Method of Fluxions, and Analysis by Infinite Series*, translated into English from the original Latin; to which is added, a Perpetual Commentary, by the translator Mr. John Colson; 1736, 4to.

17. *Several Miscellaneous Pieces, and Letters*, as follow: (1). A Letter to Mr. Boyle upon the subject of the Philosopher's Stone; inserted in the General Dictionary, under the article BOYLE.—(2). A Letter to Mr. Aston, containing directions for his Travels; ibid. under our author's article.—(3). An English Translation of a Latin Dissertation upon the Sacred Cubit of the Jews. Inserted among the miscellaneous works of Mr. John Greaves, vol. 2, published by Dr. Thomas Birch, in 1737, 2 vols. 8vo. This Dissertation was found subjoined to a work of Sir Isaac's, not finished, entitled *Lexicon Propheticum*.—(4). Four Letters from Sir Isaac Newton to Dr. Bentley, containing some arguments in proof of a Deity; 1756, 8vo.—(5). Two Letters to Mr. Clarke, &c.

18. *Observations on the Prophecies of Daniel and the Apocalypse of St. John*; 1733, 4to.

19. *Is. Newtoni Elementa Perspectivæ Universalis*; 1746, 8vo.

20. *Tables for purchasing College Leases*; 1742, 12mo.

21. Corollaries, by Whiston.

22. A collection of several pieces of our author's, under the following title, *Newtoni Is. Opuscula Mathematica Philos. & Philol.* collegit J. Castilioneus; Laus. 1744, 4to. eight tomes.

23. *Two Treatises* of the Quadrature of Curves, and Analysis by Equations of an Infinite Number of Terms, ex-

plained: tranflated by John Stewart, with a large Commentary; 1745, 4to.

24. *Defcription of an Inftrument* for obferving the Moon's diftance from the fixed ftars at fea. Philof. Tranf. vol. 42.

25. Newton alfo publifhed *Barrow's Optical Lectures* in 1699, 4to; and *Bern. Varenii Geographia, &c.*; 1681, 8vo.

26. The whole works of Newton, publifhed by Dr. Horfley; 1779, 4to. in 5 vols.

The following is a lift of the papers left by Newton at his death, as mentioned above.

A Catalogue of Sir Ifaac Newton's Manufcripts and Papers, as annexed to a bond, given by Mr. Conduit, to the adminiftrators of Sir Ifaac; by which he obliges himfelf to account for any profit he fhall make by publifhing any of the papers.

Dr. Pellet, by agreement of the executors, entered into Acts of the Prerogative Court, being appointed to perufe all the papers, and judge which were proper for the prefs.

1. Viaticum Nautarum; by Robert Wright.
2. Mifcellanea; not in Sir Ifaac's hand-writing.
3. Mifcellanea; part in Sir Ifaac's hand.
4. Trigonometria; about 5 fheets.
5. Definitions.
6. Mifcellanea; part in Sir Ifaac's hand.
7. 40 fheets in 4to, relating to Church Hiftory.
8. 126 fheets written on one fide, being foul draughts of the Prophetic Style.
9. 88 fheets relating to Church Hiftory.
10. About 70 loofe fheets in fmall 4to, of Chemical papers; fome of which are not in Sir Ifaac's hand.
11. About 62 ditto, in folio.
12. About 15 large fheets, doubled into 4to; Chemical.
13. About 8 fheets ditto, written on one fide.
14. About 5 fheets of foul papers, relating to Chemiftry.
15. 12 half-fheets of ditto.
16. 104 half-fheets in 4to, ditto.
17. About 22 fheets in 4to, ditto.

18. 24 ſheets in 4to, upon the Prophecies.

19. 29 half-ſheets, being an anſwer to Mr. Hook, on Sir Iſaac's Theory of Colours.

20. 87 half-ſheets relating to the Optics, ſome of which are not in Sir Iſaac's hand.

From No. 1 to No. 20 examined on the 20th of May, 1727, and judged not fit to be printed.

T. Pellet.

Witneſs, *Tho. Pilkington.*

21. 328 half-ſheets, in folio, and 63 in ſmall 4to; being looſe and foul papers relating to the Revelations and Prophecies.

22. 8 half-ſheets in ſmall 4to, relating to Church Matters.

23. 24 half-ſheets in ſmall 4to; being a Diſcourſe relating to the 2d of Kings.

24. 353 half-ſheets in folio, and 57 in ſmall 4to; being foul and looſe papers relating to Figures and Mathematics.

25. 201 half-ſheets in folio, and 21 in ſmall 4to; looſe and foul papers relating to the Commercium Epiſtolicum.

26. 91 half-ſheets in ſmall 4to, in Latin, upon the Temple of Solomon.

27. 37 half-ſheets in folio, upon the Hoſt of Heaven, the Sanctuary, and other Church Matters.

28. 44 half-ſheets in folio, upon ditto.

29. 25 half-ſheets in folio; being a farther account of the Hoſt of Heaven.

30. 51 half-ſheets in folio; being an Hiſtorical Account of two notable Corruptions of Scripture.

31. 88 half-ſheets in ſmall 4to; being Extracts of Church Hiſtory.

32. 116 half-ſheets in folio; being Paradoxical Queſtions concerning Athanaſius, of which ſeveral leaves in the beginning are very much damaged.

33. 56 half-ſheets in folio, De Motu Corporum; the greateſt part not in Sir Iſaac's hand.

34. 61 half-sheets in small 4to; being various sections on the Apocalypse.

35. 25 half-sheets in folio, of the Working of the Mystery of Iniquity.

36. 20 half-sheets in folio, of the Theology of the Heathens.

37. 24 half-sheets in folio; being an Account of the Contest between the Host of Heaven and the Transgressors of the Covenant.

38. 31 half-sheets in folio; being Paradoxical Questions concerning Athanasius.

39. 107 quarter-sheets in small 4to, upon the Revelations.

40. 174 half-sheets in folio; being loose papers relating to Church History.

May 22, 1727, examined from No. 21 to No. 40 inclusive, and judged them not fit to be printed: only No. 33 and No. 38 should be reconsidered.

T. Pellet.

Witness, *Tho. Pilkington.*

41. 167 half-sheets in folio; being loose and foul papers relating to the Commercium Epistolicum.

42. 21 half-sheets in folio; being the 3d letter upon Texts of Scripture: very much damaged.

43. 31 half-sheets in folio; being foul papers relating to Church Matters.

44. 495 half-sheets in folio; being loose and foul papers relating to Calculations and Mathematics.

45. 335 half-sheets in folio; being loose and foul papers relating to the Chronology.

46. 112 sheets in small 4to, relating to the Revelations, and other Church Matters.

47. 126 half-sheets in folio; being loose papers relating to the Chronology, part in English and part in Latin.

48. 400 half-sheets in folio; being loose Mathematical papers.

49. 109 sheets in 4to, relating to the Prophecies, and Church Matters.

50. 127 half-sheets in folio, relating to the University; great part not in Sir Isaac's hand.
51. 18 sheets in 4to; being Chemical papers.
52. 255 quarter-sheets; being Chemical papers.
53. An Account of Corruptions of Scripture; not in Sir Isaac's hand.
54. 31 quarter-sheets; being Flammell's Explication of Hieroglyphical Figures.
55. About 350 half-sheets; being Miscellaneous papers.
56. 6 half-sheets; being an Account of the Empires, &c. represented by St. John.
57. 9 half-sheets folio, and 71 quarter-sheets 4to; being Mathematical papers.
58. 140 half-sheets, in 9 chapters, and 2 pieces in folio, titled, Concerning the Language of the Prophets.
59. 606 half-sheets folio, relating to the Chronology; 9 more in Latin.
60. 182 half sheets folio; being loose papers relating to the Chronology and Prophecies.
61. 144 quarter-sheets, and 95 half-sheets folio; being loose Mathematical papers.
62. 137 half-sheets folio; being loose papers relating to the Dispute with Leibnitz.
63. A folio Common-place book; part in Sir Isaac's hand.
64. A bundle of English Letters to Sir Isaac, relating to Mathematics.
65. 54 half-sheets; being loose papers found in the Principia.
66. A bundle of loose Mathematical Papers; not Sir Isaac's.
67. A bundle of French and Latin Letters to Sir Isaac.
68. 136 sheets folio, relating to Optics.
69. 22 half-sheets folio, De Rationibus Motuum, &c.; not in Sir Isaac's hand.
70. 70 half-sheets folio; being loose Mathematical papers.

71. 38 half-sheets folio; being loose papers relating to Optics.

72. 47 half-sheets folio; being loose papers relating to Chronology and Prophecies.

73. 40 half-sheets folio; Processus Mysterii Magni Philosophicus, by Wm. Yworth; not in Sir Isaac's hand.

74. 5 half-sheets; being a letter from Rizzetto to Martine, in Sir Isaac's hand.

75. 41 half-sheets; being loose papers of several kinds, part in Sir Isaac's hand.

76. 40 half-sheets; being loose papers, foul and dirty, relating to Calculations.

77. 90 half-sheets folio; being loose Mathematical papers.

78. 176 half-sheets folio; being loose papers relating to Chronology.

79. 176 half-sheets folio; being loose papers relating to the Prophecies.

80. { 12 half-sheets folio; an Abstract of the Chronology.
92 half-sheets folio; the Chronology. }

81. 40 half-sheets folio; the History of the Prophecies, in 10 chapters, and part of the 11th unfinished.

82. 5 small bound books in 12mo; the greatest part not in Sir Isaac's hand, being rough Calculations.

May 26th, 1727, examined from No. 41 to No. 82 inclusive, and judged not fit to be printed, except No. 80, which is agreed to be printed, and part of No. 61 and 81, which are to be reconsidered.

T. Pellet.

Witness, *Tho. Pilkington.*

It is astonishing what care and industry Sir Isaac had employed about the papers relating to Chronology, Church History, &c.; as, on examining the papers themselves, which are in the possession of the family of the earl of Portsmouth, it appears that many of them are copies over

and over again, often with little or no variation; the whole number being upwards of 4000 ſheets in folio, or 8 reams of folio paper; beſide the bound books, &c. in this catalogue, of which the number of ſheets is not mentioned.— Of theſe there have been publiſhed only the Chronology, and Obſervations on the Prophecies of Daniel, and the Apocalypſe of St. John.

THE

MATHEMATICAL PRINCIPLES

OF

NATURAL PHILOSOPHY.

Definitions.

DEFINITION I.

The quantity of matter is the measure of the same, arising from its density and bulk conjunctly.

THUS air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction; and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter every where under the name of Body or Mass. And the same is known by the weight of each body: for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shewn hereafter.

DEFINITION II.

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly.

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

DEFINITION III.

The vis insita, *or innate force of matter, is a power of resisting, by which every body, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line.*

This force is ever proportional to the body whose force it is; and differs nothing from the inactivity of the mass, but in our manner of conceiving it. A body from the inactivity of matter, is not without difficulty put out of its state of rest or motion. Upon which account, this *vis insita,* may, by a most significant name, be called *vis inertiæ,* or force of inactivity. But a body exerts this force only, when another force, impressed upon it, endeavours to change its condition; and the exercise of this force may be considered both as resistance and impulse: it is resistance, in so far as the body, for maintaining its present state, withstands the force impressed; it is impulse, in so far as the body, by not easily giving way to the impressed force of another, endeavours to change the state of that other. Resistance is usually ascribed to bodies at rest, and impulse to those in motion: but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so.

DEFINITION IV.

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.

This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis inertiæ* only. Impressed forces are of different origins; as from percussion, from pressure, from centripetal force.

DEFINITION V.

A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the centre of the earth; magnetism, by which iron tends to the loadstone; and that force, whatever it is, by which the planets are perpetually drawn aside from the rectilinear motions,

which otherwife they would purfue, and made to revolve in curvilinear orbits. A ftone, whirled about in a fling, endeavours to recede from the hand that turns it; and by that endeavour, diftends the fling, and that with fo much the greater force, as it is revolved with the greater velocity, and as foon as ever it is let go, flies away. That force which oppofes itfelf to this endeavour, and by which the fling perpetually draws back the ftone towards the hand, and retains it in its orbit, becaufe it is directed to the hand as the centre of the orbit, I call the centripetal force. And the fame thing is to be underftood of all bodies, revolved in any orbits. They all endeavour to recede from the centres of their orbits; and were it not for the oppofition of a contrary force which reftrains them to, and detains them in their orbits, which I therefore call centripetal, would fly off in right lines, with an uniform motion. A projectile, if it was not for the force of gravity, would not deviate towards the earth, but would go off from it in a right line, and that with an uniform motion, if the refiftance of the air was taken away. It is by its gravity that it is drawn afide perpetually from its rectilinear courfe, and made to deviate towards the earth, more or lefs, according to the force of its gravity, and the velocity of its motion. The lefs its gravity is, for the quantity of its matter, or the greater the velocity with which it is projected, the lefs will it deviate from a rectilinear courfe, and the farther it will go. If a leaden ball, projected from the top of a mountain by the force of gun-powder with a given velocity, and in a direction parallel to the horizon, is carried in a curve line to the diftance of two miles before it falls to the ground; the fame, if the refiftance of the air was took away, with a double or decuple velocity, would fly twice or ten times as far. And by increafing the velocity, we may at pleafure increafe the diftance to which it might be projected, and diminifh the curvature of the line, which it might defcribe, till at laft it fhould fall at the diftance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or laftly, fo that it might never fall to the earth, but go forwards into the celeftial fpaces, and proceed in its motion *in infinitum*. And after the fame manner that a projectile, by the force of gra-

vity, may be made to revolve in an orbit, and go round the whole earth, the moon alſo, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be perpetually drawn aſide towards the earth, out of the rectilinear way, which by its innate force it would purſue; and be made to revolve in the orbit which it now deſcribes: nor could the moon without ſome ſuch force, be retained in its orbit. If this force was too ſmall, it would not ſufficiently turn the moon out of a rectilinear courſe: if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is neceſſary, that the force be of a juſt quantity, and it belongs to the mathematicians to find the force, that may ſerve exactly to retain a body in a given orbit, with a given velocity; and *vice verſa*, to determine the curvilinear way, into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force.

The quantity of any centripetal force may be conſidered as of three kinds; abſolute, accelerative, and motive.

DEFINITION VI.

The abſolute quantity of a centripetal force is the meaſure of the ſame, proportional to the efficacy of the cauſe that propagates it from the centre, through the ſpaces round about.

Thus the magnetic force is greater in one load-ſtone and leſs in another, according to their ſizes and ſtrength.

DEFINITION VII.

The accelerative quantity of a centripetal force is the meaſure of the ſame, proportional to the velocity which it generates in a given time.

Thus the force of the ſame load-ſtone is greater at a leſs diſtance, and leſs at a greater: alſo the force of gravity is greater in valleys, leſs on tops of exceeding high mountains; and yet leſs (as ſhall be hereafter ſhewn) at greater diſtances from the body of the earth; but at equal diſtances, it is the ſame every where; becauſe (taking away, or allowing for, the reſiſtance of the air) it equally accelerates all falling bodies, whether heavy or light, great or ſmall.

DEFINITION VIII.

The motive quantity of a centripetal force, is the meaſure of the ſame, proportional to the motion which it generates in a given time.

Thus the weight is greater in a greater body, leſs in a leſs body; it is greater near to the earth, and leſs at remoter diſtances. This ſort of quantity is the centripetency, or propenſion of the whole body towards the centre, or, as I may ſay, its weight; and it is ever known by the quantity of a force equal and contrary to it, that is juſt ſufficient to hinder the deſcent of the body.

Theſe quantities of forces, we may for brevity's ſake call by the names of motive, accelerative, and abſolute forces; and for diſtinction's ſake conſider them, with reſpect to the bodies that tend to the centre; to the places of thoſe bodies; and to the centre of force towards which they tend: that is to ſay, I refer the motive force to the body, as an endeavour and propenſity of the whole towards a centre, ariſing from the propenſities of the ſeveral parts taken together; the accelerative force to the place of the body, as a certain power or energy diffuſed from the centre to all places around to move the bodies that are in them; and the abſolute force to the centre, as endued with ſome cauſe, without which thoſe motive forces would not be propagated through the ſpaces round about; whether that cauſe is ſome central body (ſuch as is the load-ſtone, in the centre of the force of magnetiſm, or the earth in the centre of the gravitating force), or any thing elſe that does not yet appear. For I here deſign only to give a mathematical notion of thoſe forces, without conſidering their phyſical cauſes and ſeats.

Wherefore the accelerative force will ſtand in the ſame relation to the motive, as celerity does to motion. For the quantity of motion ariſes from the celerity drawn into the quantity of matter; and the motive force ariſes from the accelerative force drawn into the ſame quantity of matter. For the ſum of the actions of the accelerative force, upon the ſeveral particles of the body, is the motive force of the whole. Hence it is, that near the ſurface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies

is the ſame, the motive gravity or the weight is as the body: but if we ſhould aſcend to higher regions, where the accelerative gravity is leſs, the weight would be likewiſe diminiſhed, and would always be as the product of the body, by the accelerative gravity. So in thoſe regions, where the accelerative gravity is diminiſhed into one half, the Weight of a body two or three times leſs, will be four or ſix times leſs.

I likewiſe call attractions and impulſes, in the ſame ſenſe, accelerative, and motive; and uſe the words attraction, impulſe or propenſity of any ſort towards a centre, promiſcuouſly, and indifferently, one for another; conſidering thoſe forces not phyſically, but mathematically: wherefore, the reader is not to imagine, that by thoſe words, I any where take upon me to define the kind, or the manner of any action, the cauſes or the phyſical reaſon thereof, or that I attribute forces, in a true and phyſical ſenſe, to certain centres (which are only mathematical points); when at any time I happen to ſpeak of centres as attracting, or as endued with attractive powers.

SCHOLIUM.

Hitherto I have laid down the definitions of ſuch words as are leſs known, and explained the ſenſe in which I would have them to be underſtood in the following diſcourſe. I do not define time, ſpace, place and motion, as being well known to all. Only I muſt obſerve, that the vulgar conceive thoſe quantities under no other notions but from the relation they bear to ſenſible objects. And thence ariſe certain prejudices, for the removing of which, it will be convenient to diſtinguiſh them into abſolute and relative, true and apparent, mathematical and common.

I. Abſolute, true, and mathematical time, of itſelf, and from its own nature flows equably without regard to any thing external, and by another name is called duration: relative, apparent, and common time, is ſome ſenſible and external (whether accurate or unequable) meaſure of duration by the means of motion, which is commonly uſed inſtead of true time; ſuch as an hour, a day, a month, a year.

II. Abſolute ſpace, in its own nature, without regard to any thing external, remains always ſimilar and immovable.

Relative ſpace is ſome moveable dimenſion or meaſure of the abſolute ſpaces; which our ſenſes determine by its poſition to bodies; and which is vulgarly taken for immovable ſpace; ſuch is the dimenſion of a ſubterraneous, an æreal, or celeſtial ſpace, determined by its poſition in reſpect of the earth. Abſolute and relative ſpace, are the ſame in figure and magnitude; but they do not remain always numerically the ſame. For if the earth, for inſtance, moves, a ſpace of our air, which relatively and in reſpect of the earth remains always the ſame, will at one time be one part of the abſolute ſpace into which the air paſſes; at another time it will be another part of the ſame, and ſo, abſolutely underſtood, it will be perpetually mutable.

III. Place is a part of ſpace which a body takes up, and is according to the ſpace, either abſolute or relative. I ſay, a part of ſpace; not the ſituation, nor the external ſurface of the body. For the places of equal ſolids, are always equal; but their ſuperficies, by reaſon of their diſſimilar figures, are often unequal. Poſitions properly have no quantity, nor are they ſo much the places themſelves, as the properties of places. The motion of the whole is the ſame thing with the ſum of the motions of the parts; that is, the tranſlation of the whole, out of its place, is the ſame thing with the ſum of the tranſlations of the parts out of their places; and therefore the place of the whole, is the ſame thing with the ſum of the places of the parts, and for that reaſon, it is internal, and in the whole body.

IV. Abſolute motion, is the tranſlation of a body from one abſolute place into another; and relative motion, the tranſlation from one relative place into another. Thus in a ſhip under ſail, the relative place of a body is that part of the ſhip which the body poſſeſſes; or that part of its cavity which the body fills, and which therefore moves together with the ſhip: and relative reſt, is the continuance of the body in the ſame part of the ſhip, or of its cavity. But real, abſolute reſt, is the continuance of the body in the ſame part of that immovable ſpace, in which the ſhip itſelf, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at reſt, the body, which relatively reſts in the ſhip,

will really and abſolutely move with the ſame velocity which the ſhip has on the earth. But if the earth alſo moves, the true and abſolute motion of the body will ariſe, partly from the true motion of the earth, in immovable ſpace; partly from the relative motion of the ſhip on the earth: and if the body moves alſo relatively in the ſhip; its true motion will ariſe, partly from the true motion of the earth, in immovable ſpace, and partly from the relative motions as well of the ſhip on the earth, as of the body in the ſhip; and from theſe relative motions will ariſe the relative motion of the body on the earth. As if that part of the earth where the ſhip is, was truly moved toward the eaſt, with a velocity of 10010 parts; while the ſhip itſelf with a freſh gale, and full ſails, is carried towards the weſt, with a velocity expreſſed by 10 of thoſe parts; but a ſailor walks in the ſhip towards the eaſt, with 1 part of the ſaid velocity: then the ſailor will be moved truly and abſolutely in immovable ſpace towards the eaſt with a velocity of 10001 parts, and relatively on the earth towards the weſt, with a velocity of 9 of thoſe parts.

Abſolute time, in aſtronomy, is diſtinguiſhed from relative, by the equation or correction of the vulgar time. For the natural days are truly unequal, though they are commonly conſidered as equal, and uſed for a meaſure of time: aſtronomers correct this inequality for their more accurate deducing of the celeſtial motions. It may be, that there is no ſuch thing as an equable motion, whereby time may be accurately meaſured. All motions may be accelerated and retarded, but the true, or equable, progreſs of abſolute time is liable to no change. The duration or perſeverance of the exiſtence of things remains the ſame, whether the motions are ſwift or ſlow, or none at all: and therefore it ought to be diſtinguiſhed from what are only ſenſible meaſures thereof; and out of which we collect it, by means of the aſtronomical equation. The neceſſity of which equation, for determining the times of a phænomenon, is evinced as well from the experiments of the pendulum clock, as by eclipſes of the ſatellites of *Jupiter*.

As the order of the parts of time is immutable, ſo alſo is the order of the parts of ſpace. Suppoſe thoſe parts to be

moved out of their places, and they will be moved (if the expreſſion may be allowed) out of themſelves. For times and ſpaces are, as it were, the places as well of themſelves as of all other things. All things are placed in time as to order of ſucceſſion; and in ſpace as to order of ſituation. It is from their eſſence or nature that they are places; and that the primary places of things ſhould be moveable, is abſurd. Theſe are therefore the abſolute places; and tranſlations out of thoſe places, are the only abſolute motions.

But becauſe the parts of ſpace cannot be ſeen, or diſtinguiſhed from one another by our ſenſes, therefore in their ſtead we uſe ſenſible meaſures of them. For from the poſitions and diſtances of things from anybody conſidered as immovable, we define all places: and then with reſpect to ſuch places, we eſtimate all motions, conſidering bodies as tranſferred from ſome of thoſe places into others. And ſo inſtead of abſolute places and motions we uſe relative ones; and that without any inconvenience in common affairs: but in philoſophical diſquiſitions, we ought to abſtract from our ſenſes, and conſider things themſelves, diſtinct from what are only ſenſible meaſures of them. For it may be that there is no body really at reſt, to which the places and motions of others may be referred.

But we may diſtinguiſh reſt and motion, abſolute and relative, one from the other by their properties, cauſes and effects. It is a property of reſt, that bodies really at reſt do reſt in reſpect of one another. And therefore as it is poſſible, that in the remote regions of the fixed ſtars, or perhaps far beyond them, there may be ſome body abſolutely at reſt; but impoſſible to know, from the poſition of bodies to one another in our regions, whether any of theſe do keep the ſame poſition to that remote body; it follows that abſolute reſt cannot be determined from the poſition of bodies in our regions.

It is a property of motion, that the parts, which retain given poſitions to their wholes, do partake of the motions of thoſe wholes. For all the parts of revolving bodies endeavour to recede from the axis of motion; and the impetus of bodies moving forwards, ariſes from the joint impetus of all the parts. Therefore, if ſurrounding bodies are moved, thoſe that are re-

latively at reft within them, will partake of their motion. Upon which account, the true and abfolute motion of a body cannot be determined by the tranflation of it from thofe which only feem to reft: for the external bodies ought not only to appear at reft, but to be really at reft. For otherwife, all included bodies, befide their tranflation from near the furrounding ones, partake likewife of their true motions; and though that tranflation was not made they would not be really at reft, but only feem to be fo. For the furrounding bodies ftand in the like relation to the furrounded as the exterior part of a whole does to the interior, or as the fhell does to the kernel; but, if the fhell moves, the kernel will alfo move, as being part of the whole, without any removal from near the fhell.

A property near akin to the preceding, is this, that if a place is moved, whatever is placed therein moves along with it; and therefore a body, which is moved from a place in motion, partakes alfo of the motion of its place. Upon which account all motions from places in motion, are no other than parts of entire and abfolute motions; and every entire motion is compofed out of the motion of the body out of its firft place, and the motion of this place out of its place; and fo on, until we come to fome immovable place, as in the before-mentioned example of the failor. Wherefore entire and abfolute motions can be no otherwife determined than by immovable places; and for that reafon I did before refer thofe abfolute motions to immovable places, but relative ones to moveable places. Now no other places are immovable but thofe that, from infinity to infinity, do all retain the fame given pofitions one to another; and upon this account muft ever remain unmoved; and do thereby conftitute, what I call, immovable fpace.

The caufes by which true and relative motions are diftinguifhed, one from the other, are the forces impreffed upon bodies to generate motion. True motion is neither generated nor altered, but by fome force impreffed upon the body moved: but relative motion may be generated or altered without any force impreffed upon the body. For it is fufficient only to imprefs fome force on other bodies with which the former is compared, that by their giving way, that relation

may be changed, in which the relative reſt or motion of this other body did conſiſt. Again, true motion ſuffers always ſome change from any force impreſſed upon the moving body; but relative motion does not neceſſarily undergo any change by ſuch forces. For if the ſame forces are likewiſe impreſſed on thoſe other bodies, with which the compariſon is made, that the relative poſition may be preſerved, then that condition will be preſerved in which the relative motion conſiſts. And therefore any relative motion may be changed when the true motion remains unaltered, and the relative may be preſerved when the true ſuffers ſome change. Upon which accounts, true motion does by no means conſiſt in ſuch relations.

The effects which diſtinguiſh abſolute from relative motion are, the forces of receding from the axis of circular motion. For there are no ſuch forces in a circular motion purely relative, but in a true and abſolute circular motion, they are greater or leſs, according to the quantity of the motion. If a veſſel, hung by a long cord, is ſo often turned about that the cord is ſtrongly twiſted, then filled with water, and held at reſt together with the water; after, by the ſudden action of another force, it is whirled about the contrary way, and while the cord is untwiſting itſelf, the veſſel continues for ſome time in this motion; the ſurface of the water will at firſt be plain, as before the veſſel began to move: but the veſſel, by gradually communicating its motion to the water, will make it begin ſenſibly to revolve, and recede by little and little from the middle, and aſcend to the ſides of the veſſel, forming itſelf into a concave figure (as I have experienced), and the ſwifter the motion becomes, the higher will the water riſe, till at laſt, performing its revolutions in the ſame times with the veſſel, it becomes relatively at reſt in it. This aſcent of the water ſhews its endeavour to recede from the axis of its motion; and the true and abſolute circular motion of the water, which is here directly contrary to the relative, diſcovers it ſelf, and may be meaſured by this endeavour. At firſt, when the relative motion of the water in the veſſel was greateſt, it produced no endeavour to recede from the axis: the water ſhewed no tendency to the circumference, nor any aſcent

towards the fides of the veffel, but remained of a plain furface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreafed, the afcent thereof towards the fides of the veffel proved its endeavour to recede from the axis; and this endeavour fhewed the real circular motion of the water perpetually increafing, till it had acquired its greateft quantity, when the water refted relatively in the veffel. And therefore this endeavour does not depend upon any tranflation of the water in refpect of the ambient bodies, nor can true circular motion be defined by fuch tranflations. There is only one real circular motion of any one revolving body, correfponding to only one power of endeavouring to recede from its axis of motion, as its proper and edequate effect: but relative motions in one and the fame body are innumerable, according to the various relations it bears to external bodies, and like other relations, are altogether deftitute of any real effect, any otherwife than they may perhaps participate of that one only true motion. And therefore in their fyftem who fuppofe that our heavens, revolving below the fphere of the fixed ftars, carry the planets along with them; the feveral parts of thofe heavens, and the planets, which are indeed relatively at reft in their heavens, do yet really move. For they change their pofition one to another (which never happens to bodies truly at reft), and being carried together with their heavens, participate of their motions, and as parts of revolving wholes, endeavour to recede from the axis of their motions.

Wherefore relative quantities are not the quantities themfelves, whofe names they bear, but thofe fenfible meafures of them (either accurate or inaccurate) which are commonly ufed inftead of the meafured quantities themfelves. And if the meaning of words is to be determined by their ufe, then by the names time, fpace, place and motion, their meafures are properly to be underftood; and the expreffion will be unufual, and purely mathematical, if the meafured quantities themfelves are meant. Upon which account, they do ftrain the facred writings, who there interpret thofe words for the meafured quantities. Nor do thofe lefs defile the purity of mathematical and philofophical truths, who confound real quantities themfelves with their relations and vulgar meafures.

It is indeed a matter of great difficulty to difcover, and effectually to diftinguifh, the true motions of particular bodies from the apparent: becaufe the parts of that immovable fpace in which thofe motions are performed, do by no means come under the obfervation of our fenfes. Yet the thing is not altogether defperate; for we have fome arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the caufes and effects of the true motions. For inftance, if two globes, kept at a given diftance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tenfion of the cord, difcover the endeavour of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces fhould be impreffed at once on the alternate faces of the globes to augment or diminifh their circular motions, from the increafe or decreafe of the tenfion of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces thofe forces ought to be impreff-ed, that the motions of the globes might be moft augmented; that is, we might difcover their hindermoft faces, or thofe which, in the circular motion, do follow. But the faces which follow being known, and confequently the oppofite ones that precede, we fhould likewife know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immenfe vacuum, where there was nothing external or fenfible with which the globes could be compared. But now, if in that fpace fome remote bodies were placed that kept always a given pofition one to another, as the fixed ftars do in our regions, we could not indeed determine from the relative tranflation of the globes among thofe bodies, whether the motion did belong to the globes or to the bodies. But if we obferved the cord, and found that its tenfion was that very tenfion which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at reft; and then, laftly, from the tranflation of the globes

among the bodies, we ſhould find the determination of their motions. But how we are to collect the true motions from their cauſes, effects, and apparent differences; and, *vice verſa*, how from the motions, either true or apparent, we may come to the knowledge of their cauſes and effects, ſhall be explained more at large in the following Tract. For to this end it was that I compoſed it.

AXIOMS; OR LAWS OF MOTION.

LAW I.

Every body perſeveres in its ſtate of reſt, or of uniform motion in a right line, unleſs it is compelled to change that ſtate by forces impreſſed thereon.

PROJECTILES perſevere in their motions, ſo far as they are not retarded by the reſiſtance of the air, or impelled downwards by the force of gravity. A top, whoſe parts by their coheſion are perpetually drawn aſide from rectilinear motions, does not ceaſe its rotation, otherwiſe than as it is retarded by the air. The greater bodies of the planets and comets, meeting with leſs reſiſtance in more free ſpaces, preſerve their motions both progreſſive and circular for a much longer time.

LAW II.

The alteration of motion is ever proportional to the motive force impreſſed; and is made in the direction of the right line in which that force is impreſſed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impreſſed altogether and at once, or gradually and ſucceſſively. And this motion (being always directed the ſame way with the generating force), if the body moved before, is added to or ſubducted from the former motion, according as they directly conſpire with or are directly contrary to each other; or obliquely joined, when they are oblique, ſo as to produce a new motion compounded from the determination of both.

LAW III.

To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities, but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are reciprocally proportional to the bodies. This law takes place also in attractions, as will be proved in the next scholium.

COROLLARY I.

A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart. (Pl. 1. Fig. 1.)

If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B; and by the force N impressed apart in the same place, should be carried from A to C; complete the parallelogram ABCD, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed

or not; and therefore at the end of that time it will be found ſomewhere in the line BD. By the ſame argument, at the end of the ſame time it will be found ſomewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law I.

COROLLARY II.

And hence is explained the compoſition of any one direct force AD, *out of any two oblique forces* AB *and* BD; *and, on the contrary, the reſolution of any one direct force* AD *into two oblique forces* AB *and* BD: *which compoſition and reſolution are abundantly confirmed from mechanics.* (Fig. 2.)

As if the unequal radii OM and ON drawn from the centre O of any wheel, ſhould ſuſtain the weights A and P by the cords MA and NP; and the forces of thoſe weights to move the wheel were required. Through the centre O draw the right line KOL, meeting the cords perpendicularly in K and L; and from the centre O, with OL the greater of the diſtances OK and OL, deſcribe a circle, meeting the cord MA in D: and drawing OD, make AC parallel and DC perpendicular thereto. Now, it being indifferent whether the points K, L, D, of the cords be fixed to the plane of the wheel or not, the weights will have the ſame effect whether they are ſuſpended from the points K and L, or from D and L. Let the whole force of the weight A be repreſented by the line AD, and let it be reſolved into the forces AC and CD; of which the force AC, drawing the radius OD directly from the centre, will have no effect to move the wheel: but the other force DC, drawing the radius DO perpendicularly, will have the ſame effect as if it drew perpendicularly the radius OL equal to OD; that is, it will have the ſame effect as the weight P, if that weight is to the weight A as the force DC is to the force DA; that is (becauſe of the ſimilar triangles ADC, DOK), as OK to OD or OL. Therefore the weights A and P, which are reciprocally as the radii OK and OL that lie in the ſame right line, will be equipollent, and ſo remain in equilibrio: which is the well known property of the balance, the lever, and the wheel. If either weight is greater than in this ratio, its force to move the wheel will be ſo much the greater.

If the weight p, equal to the weight P, is partly ſuſpended by the cord Np, partly ſuſtained by the oblique plane pG; draw pH, NH, the former perpendicular to the horizon, the latter to the plane pG; and if the force of the weight p tending downwards is repreſented by the line pH, it may be reſolved into the forces pN, HN. If there was any plane perpendicular to the cord pN, cutting the other plane pG in a line parallel to the horizon, and the weight p was ſupported only by thoſe planes pQ, pG, it would preſs thoſe planes perpendicularly with the forces pN, HN; to wit, the plane pQ with the force pN, and the plane pG with the force HN. And therefore if the plane pQ was taken away, ſo that the weight might ſtretch the cord, becauſe the cord, now ſuſtaining the weight, ſupplies the place of the plane that was removed, it will be ſtrained by the ſame force pN which preſſed upon the plane before. Therefore the tenſion of this oblique cord pN will be to that of the other perpendicular cord PN as pN to pH. And therefore if the weight p is to the weight A in a ratio compounded of the reciprocal ratio of the leaſt diſtances of the cords pN, AM, from the centre of the wheel, and of the direct ratio of pH to pN, the weights will have the ſame effect towards moving the wheel, and will therefore ſuſtain each other; as any one may find by experiment.

But the weight p preſſing upon thoſe two oblique planes, may be conſidered as a wedge between the two internal ſurfaces of a body ſplit by it; and hence the forces of the wedge and the mallet may be determined; for because the force with which the weight p preſſes the plane pQ is to the force with which the ſame, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line pH towards both the planes, as pN to pH; and to the force with which it preſſes the other plane pG, as pN to NH. And thus the force of the ſcrew may be deduced from a like reſolution of forces; it being no other than a wedge impelled with the force of a lever. Therefore the uſe of this Corollary ſpreads far and wide, and by that diffuſive extent the truth thereof is farther confirmed. For on what has been ſaid depends the

whole doctrine of mechanics variously demonstrated by different authors. For from hence are easily deduced the forces of machines, which are compounded of wheels, pullies, levers, cords, and weights, ascending directly or obliquely, and other mechanical powers; as also the force of the tendons to move the bones of animals.

COROLLARY III.

The quantity of motion, which is collected by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves.

For action and its opposite re-action are equal, by Law 3, and therefore, by Law 2, they produce in the motions equal changes towards opposite parts. Therefore if the motions are directed towards the same parts, whatever is added to the motion of the preceding body will be subducted from the motion of that which follows; so that the sum will be the same as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards opposite parts will remain the same.

Thus if a spherical body A with two parts of velocity is triple of a spherical body B which follows in the same right line with ten parts of velocity, the motion of A will be to that of B as 6 to 10. Suppose, then, their motions to be of 6 parts and of 10 parts, and the sum will be 16 parts. Therefore upon the meeting of the bodies, if A acquire 3, 4, or 5 parts of motion, B will lose as many; and therefore after reflexion A will proceed with 9, 10, or 11 parts, and B with 7, 6, or 5 parts; the sum remaining always of 16 parts as before. If the body A acquire 9, 10, 11, or 12 parts of motion, and therefore after meeting proceed with 15, 16, 17, or 18 parts, the body B, losing so many parts as A has got, will either proceed with one part, having lost 9, or stop and remain at rest, as having lost its whole progressive motion of 10 parts; or it will go back with one part, having not only lost its whole motion, but (if I may so say) one part more; or it will go back with 2 parts, because a progressive motion of 12

parts is taken off. And ſo the ſums of the conſpiring motions $15+1$, or $16+0$, and the differences of the contrary motions $17-1$ and $18-2$, will always be equal to 16 parts, as they were before the meeting and reflexion of the bodies. But, the motions being known with which the bodies proceed after reflexion, the velocity of either will be alſo known, by taking the velocity after to the velocity before reflexion, as the motion after is to the motion before. As in the laſt caſe, where the motion of the body A was of 6 parts before reflexion and of 18 parts after, and the velocity was of 2 parts before reflexion, the velocity thereof after reflexion will be found to be of 6 parts; by ſaying, as the 6 parts of motion before to 18 parts after, ſo are 2 parts of velocity before reflexion to 6 parts after.

But if the bodies are either not ſpherical, or, moving in different right lines, impinge obliquely one upon the other, and their motions after reflexion are required, in thoſe caſes we are firſt to determine the poſition of the plane that touches the concurring bodies in the point of concourſe; then the motion of each body (by Corol. 2) is to be reſolved into two, one perpendicular to that plane, and the other parallel to it. This done, becauſe the bodies act upon each other in the direction of a line perpendicular to this plane, the parallel motions are to be retained the ſame after reflexion as before; and to the perpendicular motions we are to aſſign equal changes towards the contrary parts; in ſuch manner that the ſum of the conſpiring and the difference of the contrary motions may remain the ſame as before. From ſuch kind of reflexions alſo ſometimes ariſe the circular motions of bodies about their own centres. But theſe are caſes which I do not conſider in what follows; and it would be too tedious to demonſtrate every particular that relates to this ſubject.

COROLLARY IV.

The common centre of gravity of two or more bodies does not alter its ſtate of motion or reſt by the actions of the bodies among themſelves; and therefore the common centre of gravity of all bodies acting upon each other (excluding outward

actions and impediments) is either at rest, or moves uniformly in a right line.

For if two points proceed with an uniform motion in right lines, and their distance be divided in a given ratio, the dividing point will be either at rest, or proceed uniformly in a right line. This is demonstrated hereafter in lem. 23 and its corol. when the points are moved in the same plane; and by a like way of arguing, it may be demonstrated when the points are not moved in the same plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at rest, or proceeds uniformly in a right line; because the line which connects the centres of those two bodies so moving is divided at that common centre in a given ratio. In like manner the common centre of those two and that of a third body will be either at rest or moving uniformly in a right line; because at that centre the distance between the common centre of the two bodies, and the centre of this last, is divided in a given ratio. In like manner the common centre of these three, and of a fourth body, is either at rest, or moves uniformly in a right line; because the distance between the common centre of the three bodies, and the centre of the fourth is there also divided in a given ratio, and so on *in infinitum*. Therefore in a system of bodies where there is neither any mutual action among themselves, nor any foreign force impressed upon them from without, and which consequently move uniformly in right lines, the common centre of gravity of them all is either at rest, or moves uniformly forwards in a right line.

Moreover, in a system of two bodies mutually acting upon each other, since the distances between their centres and the common centre of gravity of both are reciprocally as the bodies, the relative motions of those bodies, whether of approaching to or of receding from that centre, will be equal among themselves. Therefore since the changes which happen to motions are equal and directed to contrary parts, the common centre of those bodies, by their mutual action between themselves, is neither promoted nor retarded, nor suffers any change as to its state of motion or rest. But in a

ſyſtem of ſeveral bodies, becauſe the common centre of gravity of any two acting mutually upon each other ſuffers no change in its ſtate by that action; and much leſs the common centre of gravity of the others with which that action does not intervene; but the diſtance between thoſe two centres is divided by the common centre of gravity of all the bodies into parts reciprocally proportional to the total ſums of thoſe bodies whoſe centres they are; and therefore while thoſe two centres retain their ſtate of motion or reſt, the common centre of all does alſo retain its ſtate: it is manifeſt that the common centre of all never ſuffers any change in the ſtate of its motion or reſt from the actions of any two bodies between themſelves. But in ſuch a ſyſtem all the actions of the bodies among themſelves either happen between two bodies, or are compoſed of actions interchanged betweeen ſome two bodies; and therefore they do never produce any alteration in the common centre of all as to its ſtate of motion or reſt. Wherefore ſince that centre, when the bodies do not act mutually one upon another, either is at reſt or moves unifotmly forward in ſome right line, it will, notwithſtanding the mutual actions of the bodies among themſelves, always perſevere in its ſtate, either of reſt, or of proceeding uniformly in a right line, unleſs it is forced out of this ſtate by the action of ſome power impreſſed from without upon the whole ſyſtem. And therefore the ſame law takes place in a ſyſtem conſiſting of many bodies as in one ſingle body, with regard to their perſevering in their ſtate of motion or of reſt. For the progreſſive motion, whether of one ſingle body, or of a whole ſyſtem of bodies, is always to be eſtimated from the motion of the centre of gravity.

COROLLARY V.

The motions of bodies included in a given ſpace are the ſame among themſelves, whether that ſpace is at reſt, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the ſame parts, and the ſums of thoſe that tend towards contrary parts, are, at firſt (by ſuppoſition), in both caſes the ſame; and it is from thoſe ſums and differences that the colliſions and impulſes

do arise with which the bodies mutually impinge one upon another. Wherefore (by Law 2) the effects of those collisions will be equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the mutual motions of the bodies among themselves in the other. A clear proof of which we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.

COROLLARY VI.

If bodies, any how moved among themselves, are urged in the direction of parallel lines by equal accelerative forces, they will all continue to move among themselves, after the same manner as if they had been urged by no such forces.

For these forces acting equally (with respect to the quantities of the bodies to be moved), and in the direction of parallel lines, will (by Law 2) move all the bodies equally (as to velocity), and therefore will never produce any change in the positions or motions of the bodies among themselves.

SCHOLIUM.

Hitherto I have laid down such principles as have been received by mathematicians, and are confirmed by abundance of experiments. By the two first Laws and the first two Corollaries, *Galileo* discovered that the descent of bodies observed the duplicate ratio of the time, and that the motion of projectiles was in the curve of a parabola; experience agreeing with both, unless so far as these motions are a little retarded by the resistance of the air. When a body is falling, the uniform force of its gravity acting equally, impresses, in equal particles of time, equal forces upon that body, and therefore generates equal velocities; and in the whole time impresses a whole force, and generates a whole velocity proportional to the time. And the spaces described in proportional times are as the velocities and the times conjunctly; that is, in a duplicate ratio of the times. And when a body is thrown upwards, its uniform gravity impresses forces and takes off velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken off, and

those heights are as the velocities and the times conjunctly, or in the duplicate ratio of the velocities. And if a body be projected in any direction, the motion arising from its projection is compounded with the motion arising from its gravity, As if the body A by its motion of projection alone (*Fig.* 3.) could describe in a given time the right line AB, and with its motion of falling alone could describe in the same time the latitude AC; complete the paralellogram ABDC, and the body by that compounded motion will at the end of the time be found in the place D; and the curve line AED, which that body describes, will be a parabola, to which the right line AB will be a tangent in A; and whose ordinate BD will be as the square of the line AB. On the same laws and corollaries depend those things which have been demonstrated concerning the times of the vibration of pendulums, and are confirmed by the daily experiments of pendulum clocks. By the same, together with the third Law, Sir *Chrift. Wren*, Dr. *Wallis*, and Mr. *Huygens*, the greatest geometers of our times, did severally determine the rules of the congress and reflexion of hard bodies, and much about the same time communicated their discoveries to the Royal Society, exactly agreeing among themselves as to those rules. Dr. *Wallis*, indeed, was something more early in the publication; then followed Sir *Christopher Wren*, and, lastly, Mr. *Huygens*. But Sir *Christopher Wren* confirmed the truth of the thing before the Royal Society by the experiment of pendulums, which Mr. *Mariotte* soon after thought fit to explain in a treatise entirely upon that subject. But to bring this experiment to an accurate agreement with the theory, we are to have a due regard as well to the resistance of the air as to the elastic force of the concurring bodies. Let the spherical bodies AB be suspended by the parallel and equal strings AC, BD (*Fig.* 4.) from the centres C, D. About these centres, with those intervals, describe the semicircles EAF, GBH, bisected by the radii CA, DB. Bring the body A to any point R of the arc EAF, and (withdrawing the body B) let it go from thence, and after one oscillation suppose it to return to the point V: then RV will be the retardation arising from the resistance of the air. Of

this RV let ST be a fourth part, ſituated in the middle, to wit, ſo as RS and TV may be equal, and RS may be to ST as 3 to 2: then will ST repreſent very nearly the retardation during the deſcent from S to A. Reſtore the body B to its place: and, ſuppoſing the body A to be let fall from the point S, the velocity thereof in the place of reflexion A, without ſenſible error, will be the ſame as if it had deſcended *in vacuo* from the point T. Upon which account this velocity may be repreſented by the chord of the arc TA. For it is a propoſition well known to geometers, that the velocity of a pendulous body in the loweſt point is as the chord of the arc which it has deſcribed in its deſcent. After reflexion, ſuppoſe the body A comes to the place s, and the body B to the place k. Withdraw the body B, and find the place v, from which if the body A, being let go, ſhould after one oſcillation return to the place r, st may be a fourth part of rv, ſo placed in the middle thereof as to leave rs equal to tv, and let the chord of the arc tA repreſent the velocity which the body A had in the place A immediately after reflexion. For t will be the true and correct place to which the body A ſhould have aſcended, if the reſiſtance of the air had been taken off. In the ſame way we are to correct the place k to which the body B aſcends, by finding the place l to which it ſhould have aſcended *in vacuo*. And thus every thing may be ſubjected to experiment, in the ſame manner as if we were really placed *in vacuo*. Theſe things being done, we are to take the product (if I may ſo ſay) of the body A, by the chord of the arc TA (which repreſents its velocity), that we may have its motion in the place A immediately before reflexion; and then by the chord of the arc tA, that we may have its motion in the place A immediately after reflexion. And ſo we are to take the product of the body B by the chord of the arc Bl, that we may have the motion of the ſame immediately after reflexion. And in like manner, when two bodies are let go together from different places, we are to find the motion of each, as well before as after reflexion; and then we may compare the motions between themſelves, and collect the effects of the reflexion. Thus trying the thing with pendulums of ten feet, in unequal as well as equal bodies, and

making the bodies to concur after a defcent through large fpaces, as of 8, 12, or 16 feet, I found always, without an error of 3 inches, that when the bodies concurred together directly, equal changes toward the contrary parts were produced in their motions, and, of confequence, that the action and re-action were always equal. As if the body A impinged upon the body B at reft with 9 parts of motion, and lofing 7, proceeded after reflexion with 2, the body B was carried backwards with thofe 7 parts. If the bodies concurred with contrary motions, A with twelve parts of motion, and B with fix, then if A receded with 2, B receded with 8; to wit, with a deduction of 14 parts of motion on each fide. For from the motion of A fubducting 12 parts, nothing will remain; but fubducting 2 parts more, a motion will be generated of 2 parts towards the contrary way; and fo, from the motion of the body B of 6 parts, fubducting 14 parts, a motion is generated of 8 parts towards the contrary way. But if the bodies were made both to move towards the fame way, A, the fwifter, with 14 parts of motion, B, the flower, with 5, and after reflexion A went on with 5, B likewife went on with 14 parts; 9 parts being transferred from A to B. And fo in other cafes. By the congrefs and collifion of bodies, the quantity of motion, collected from the fum of the motions directed towards the fame way, or from the difference of thofe that were directed towards contrary ways, was never changed. For the error of an inch or two in meafures may be eafily afcribed to the difficulty of executing every thing with accuracy. It was not eafy to let go the two pendulums fo exactly together that the bodies fhould impinge one upon the other in the lowermoft place AB; nor to mark the places s, and k, to which the bodies afcended after congrefs. Nay, and fome errors, too, might have happened from the unequal denfitity of the parts of the pendulous bodies themfelves, and from the irregularity of the texture proceeding from other caufes.

But to prevent an objection that may perhaps be alledged againft the rule, for the proof of which this experiment was made, as if this rule did fuppofe that the bodies were either abfolutely hard, or at leaft perfectly elaftic (whereas no fuch bodies are to be found in nature), I muft add, that the expe-

riments we have been deſcribing, by no means depending upon that quality of hardneſs, do ſucceed as well in ſoft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminiſh the reflexion in ſuch a certain proportion as the quantity of the elaſtic force requires. By the theory of *Wren* and *Huygens,* bodies abſolutely hard return one from another with the ſame velocity with which they meet. But this may be affirmed with more certainty of bodies perfectly elaſtic. In bodies imperfectly elaſtic the velocity of the return is to be diminiſhed together with the elaſtic force; becauſe that force (except when the parts of bodies are bruiſed by their congreſs, or ſuffer ſome ſuch extenſion as happens under the ſtrokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and ſtrongly compreſſed. For, firſt, by letting go the pendulous bodies, and meaſuring their reflexion, I determined the quantity of their elaſtic force; and then, according to this force, eſtimated the reflexions that ought to happen in other caſes of congreſs. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity with which they met as about 5 to 9. Balls of ſteel returned with almoſt the ſame velocity: thoſe of cork with a velocity ſomething leſs: but in balls of glaſs the proportion was as about 15 to 16. And thus the third law, ſo far as it regards percuſſions and reflexions, is proved by a theory exactly agreeing with experience.

In attractions, I briefly demonſtrate the thing after this manner. Suppoſe an obſtacle is interpoſed to hinder the congreſs of any two bodies A, B, mutually attracting one the other: then if either body, as A, is more attracted towards the other body B, than that other body B is towards the firſt body A, the obſtacle will be more ſtrongly urged by the preſſure of the body A than by the preſſure of the body B, and therefore will not remain in equilibrio: but the ſtronger

preſſure will prevail, and will make the ſyſtem of the two bodies, together with the obſtacle, to move directly towards the parts on which B lies; and in free ſpaces, to go forward *in infinitum* with a motion perpetually accelerated; which is abſurd, and contrary to the firſt law. For, by the firſt law, the ſyſtem ought to perſevere in its ſtate of reſt, or of moving uniformly forward in a right line; and therefore the bodies muſt equally preſs the obſtacle, and be equally attracted one by the other. I made the experiment on the loadſtone and iron. If theſe, placed apart in proper veſſels, are made to float by one another in ſtanding water, neither of them will propel the other; but, by being equally attracted, they will ſuſtain each other's preſſure, and reſt at laſt in an equilibrium.

So the gravitation betwixt the earth and its parts is mutual. Let the earth FI (*fig.* 5.) be cut by any plane EG into two parts EGF and EGI, and their weights one towards the other will be mutually equal. For if by another plane HK, parallel to the former EG, the greater part EGI is cut into two parts EGKH and HKI, whereof HKI is equal to the part EFG firſt cut off, it is evident that the middle part EGKH will have no propenſion by its proper weight towards either ſide, but will hang, as it were, and reſt in an equilibrium betwixt both. But the one extreme part HKI will with its whole weight bear upon and preſs the middle part toward the other extreme part EGF; and therefore the force with which EGI, the ſum of the parts HKI and EGKH, tends towards the third part EGF, is equal to the weight of the part HKI, that is, to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF, one towards the other, are equal, as I was to prove. And indeed if thoſe weights were not equal, the whole earth floating in the nonreſiſting æther would give way to the greater weight, and, retiring from it, would be carried off *in infinitum*.

And as thoſe bodies are equipollent in the congreſs and reflexion, whoſe velocities are reciprocally as their innate forces, ſo in the uſe of mechanic inſtruments thoſe agents are equipollent, and mutually ſuſtain each the contrary preſ-

ſure of the other, whoſe velocities, eſtimated according to the determination of the forces, are reciprocally as the forces.

So thoſe weights are of equal force to move the arms of a balance; which during the play of the balance are reciprocally as their velocities upwards and downwards: that is, if the aſcent or deſcent is direct, thoſe weights are of equal force, which are reciprocally as the diſtances of the points at which they are ſuſpended from the axis of the balance; but if they are turned aſide by the interpoſition of oblique planes, or other obſtacles, and made to aſcend or deſcend obliquely, thoſe bodies will be equipollent, which are reciprocally as the heights of their aſcent and deſcent taken according to the perpendicular; and that on account of the determination of gravity downwards.

And in like manner in the pully, or in a combination of pullies, the force of a hand drawing the rope directly, that is, to the weight, whether aſcending directly or obliquely, as the velocity of the perpendicular aſcent of the weight to the velocity of the hand that draws the rope, will ſuſtain the weight.

In clocks and ſuch like inſtruments, made up from a combination of wheels, the contrary forces that promote and impede the motion of the wheels, if they are reciprocally as the velocities of the parts of the wheel on which they are impreſſed, will mutually ſuſtain the one the other.

The force of the ſcrew to preſs a body is to the force of the hand that turns the handles by which it is moved as the circular velocity of the handle in that part where it is impelled by the hand is to the progreſſive velocity of the ſcrew towards the preſſed body.

The forces by which the wedge preſſes or drives the two parts of the wood it cleaves are to the force of the mallet upon the wedge as the progreſs of the wedge in the direction of the force impreſſed upon it by the mallet is to the velocity with which the parts of the wood yield to the wedge, in the direction of lines perpendicular to the ſides of the wedge. And the like account is to be given of all machines.

The power and uſe of machines conſiſt only in this, that by diminiſhing the velocity we may augment the force, and the contrary: from whence, in all ſorts of proper machines, we have the ſolution of this problem; *To move a given weight with a given power,* or with a given force to overcome any other given reſiſtance. For if machines are ſo contrived that the velocities of the agent and reſiſtant are reciprocally as their forces, the agent will juſt ſuſtain the reſiſtant, but with a greater diſparity of velocity will overcome it. So that if the diſparity of velocities is ſo great as to overcome all that reſiſtance which commonly ariſes either from the attrition of contiguous bodies as they ſlide by one another, or from the coheſion of continuous bodies that are to be ſeparated, or from the weights of bodies to be raiſed, the exceſs of the force remaining, after all thoſe reſiſtances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the machine as in the reſiſting body. But to treat of mechanics is not my preſent buſineſs. I was only willing to ſhew by thoſe examples the great extent and certainty of the third law of motion. For if we eſtimate the action of the agent from its force and velocity conjunctly, and likewiſe the re-action of the impediment conjunctly from the velocities of its ſeveral parts, and from the forces of reſiſtance ariſing from the attrition, coheſion, weight, and acceleration of thoſe parts, the action and re-action in the uſe of all ſorts of machines will be found always equal to one another. And ſo far as the action is propagated by the intervening inſtruments, and at laſt impreſſed upon the reſiſting body, the ultimate determination of the action will be always contrary to the determination of the re-action.

OF THE MOTION OF BODIES.

BOOK I.

SECTION I.

Of the method of firſt and laſt ratios of quantities, by the help whereof we demonſtrate the propoſitions that follow.

LEMMA I.

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal.

If you deny it, ſuppoſe them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is againſt the ſuppoſition

LEMMA II.

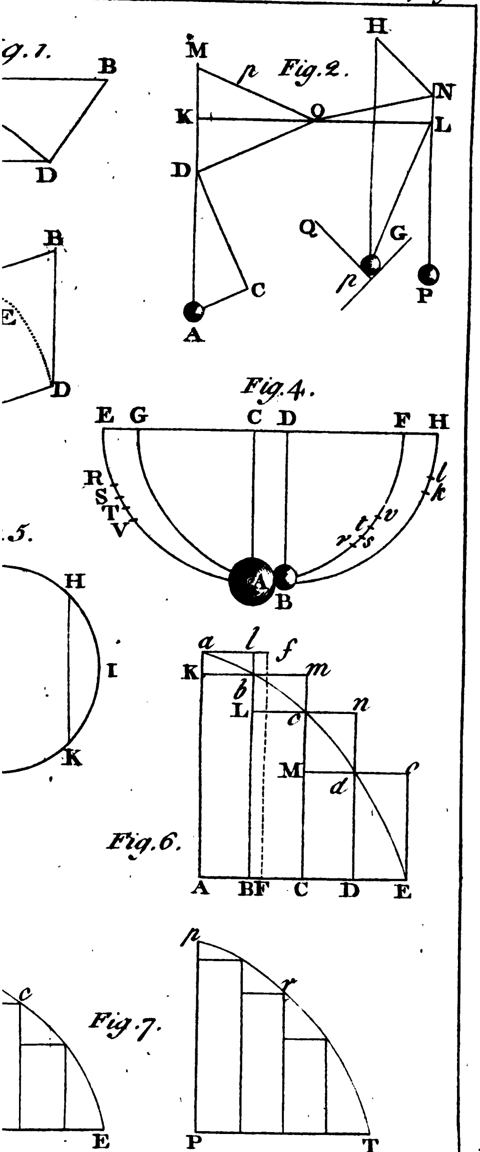
If in any figure AacE (Pl. 1. Fig. 6.), *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inſcribed any number of parallelograms* Ab, Bc, Cd, *&c. comprehended under equal baſes* AB, BC, CD, *&c. and the ſides* Bb, Cc, Dd, *&c. parallel to one ſide* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c. are completed. Then if the breadth of thoſe parallelograms be ſuppoſed to be diminiſhed, and their number to be augmented* in infinitum; *I ſay, that the ultimate ratios which the inſcribed figure* AKbLcMdD, *the circumſcribed figure* AalbmcndoE, *and curvilinear figure* AabcdE, *will have to one another, are ratios of equality.*

For the difference of the inſcribed and circumſcribed figures is the ſum of the parallelograms Kl, Lm, Mn, Do, that is (from the equality of all their baſes), the rectangle under one of their baſes Kb and the ſum of their altitudes Aa, that is, the rectangle ABla. But this rectangle, becauſe its breadth AB is ſuppoſed diminiſhed *in infinitum,* becomes leſs than any given ſpace. And therefore (by Lem. 1) the figures inſcribed and circumſcribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

LEMMA III.

The ſame ultimate ratios are alſo ratios of equality, when the breadths AB, BC, DC, *&c. of the parallelograms are unequal, and are all diminiſhed* in infinitum.

For ſuppoſe AF equal to the greateſt breadth, and complete the parallelogram FAaf. This parallelogram will be greater than the difference of the inſcribed and circumſcribed figures;



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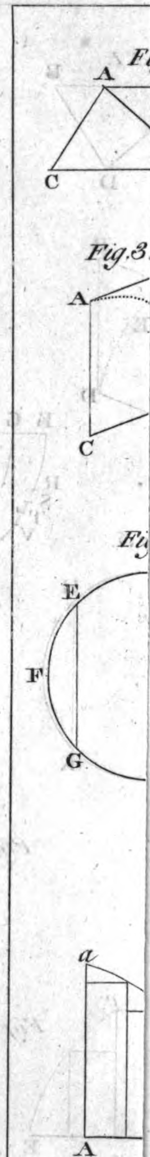


Fig.3
A
C
E
F
G
a
A

but, becaufe its breadth AF is diminifhed *in infinitum,* it will become lefs than any given rectangle. Q.E.D.

Cor. 1. Hence the ultimate fum of thofe evanefcent parallelograms will in all parts coincide with the curvilinear figure.

Cor. 2. Much more will the rectilinear figure comprehended under the chords of the evanefcent arcs ab, bc, cd, &c. ultimately coincide with the curvilinear figure.

Cor. 3. And alfo the circumfcribed rectilinear figure comprehended under the tangents of the fame arcs.

Cor. 4. And therefore thefe ultimate figures (as to their perimeters acE,) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures AacE, PprT (Pl. 1. Fig. 7.), *you infcribe (as before) two ranks of parallelograms, an equal number in each rank, and, when their breadths are diminifhed* in infinitum, *the ultimate ratios of the parallelograms in one figure to thofe in the other, each to each refpectively, are the fame; I fay, that thofe two figures* AacE, PprT, *are to one another in that fame ratio.*

For as the parallelograms in the one are feverally to the parallelograms in the other, fo (by compofition) is the fum of all in the one to the fum of all in the other; and fo is the one figure to the other; becaufe (by Lem. 3) the former figure to the former fum, and the latter figure to the latter fum, are both in the ratio of equality. Q.E.D.

Cor. Hence if two quantities of any kind are any how divided into an equal number of parts, and thofe parts, when their number is augmented, and their magnitude diminifhed *in infinitum,* have a given ratio one to the other, the firft to the firft, the fecond to the fecond, and fo on in order, the whole quantities will be one to the other in that fame given ratio. For if, in the figures of this lemma, the parallelograms are taken one to the other in the ratio of the parts, the fum of the parts will always be as the fum of the parallelograms; and therefore fuppofing the number of the parallelograms and parts to be augmented, and their magnitudes diminifhed *in infinitum,* thofe

ſums will be in the ultimate ratio of the parallelogram in the one figure to the correſpondent parallelogram in the other; that is (by the ſuppoſition), in the ultimate ratio of any part of the one quantity to the correſpondent part of the other.

LEMMA V.

In ſimilar figures, all ſorts of homologous ſides, whether curvilinear or rectilinear, are proportional; and the areas are in the duplicate ratio of the homologous ſides.

LEMMA VI.

If any arc ACB (Pl. 2. Fig. 1.) *given in poſition is ſubtended by its chord* AB, *and in any point* A, *in the middle of the continued curvature, is touched by a right line* AD, *produced both ways; then if the points* A *and* B *approach one another and meet, I ſay, the angle* BAD, *contained between the chord and the tangent, will be diminiſhed* in infinitum, *and ultimately will vaniſh.*

For if that angle does not vaniſh, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is againſt the ſuppoſition.

LEMMA VII.

The ſame things being ſuppoſed, I ſay that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality (Pl. 2. Fig. 1.).

For while the point B approaches towards the point A, conſider always AB and AD as produced to the remote points b and d, and parallel to the ſecant BD draw bd: and let the arc Acb be always ſimilar to the arc ACB. Then, ſuppoſing the points A and B to coincide, the angle dAb will vaniſh, by the preceding lemma; and therefore the right lines Ab, Ad (which are always finite), and the intermediate arc Acb, will coincide, and become equal among themſelves. Wherefore the right lines AB, AD, and the intermediate arc ACB (which are always proportional to the former), will vaniſh, and ultimately acquire the ratio of equality. Q.E.D.

Cor. 1. Whence if through B (Pl. 2. Fig. 2.) we draw BF parallel to the tangent, always cutting any right line AF paſſing through A in F, this line BF will be ultimately in the

ratio of equality with the evanescent arc ACB; because, completing the parallelogram AFBD, it is always in a ratio of equality with AD.

Cor. 2. And if through B and A more right lines are drawn, as BE, BD, AF, AG, cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissas AD, AE, BF, BG, and of the chord and arc AB, any one to any other, will be the ratio of equality.

Cor. 3. And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other.

LEMMA VIII.

If the right lines AR, BR (pl. 2. fig. 1.), *with the arc* ACB, *the chord* AB, *and the tangent* AD, *constitute three triangles* RAB, RACB, RAD, *and the points* A *and* B *approach and meet: I say, that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.*

For while the point B approaches towards the point A, consider always AB, AD, AR, as produced to the remote points b, d, and r, and rbd as drawn parallel to RD, and let the arc Acb be always similar to the arc ACB. Then supposing the points A and B to coincide, the angle bAd will vanish; and therefore the three triangles rAb, rAcb, rAd (which are always finite), will coincide, and on that account become both similar and equal. And therefore the triangles RAB, RACB, RAD, which are always similar and proportional to these, will ultimately become both similar and equal among themselves. Q.E.D.

Cor. And hence in all our reasonings about ultimate ratios, we may indifferently use any one of those triangles for any other.

LEMMA IX.

If a right line AE (pl. 2. fig. 3.), *and a curve line* ABC, *both given by position, cut each other in a given angle* A; *and to that right line, in another given angle,* BD, CE *are ordinately applied, meeting the curve in* B, C; *and the points* B *and* C *together approach towards and meet in the point* A: *I say, that the areas of the triangles* ABD, ACE, *will*

ultimately be one to the other in the duplicate ratio of the ſides.

For while the points B, C approach towards the point A, ſuppoſe always AD to be produced to the remote points d and e, ſo as Ad, Ae may be proportional to AD, AE; and the ordinates db, ec, to be drawn parallel to the ordinates DB and EC, and meeting AB and AC produced in b and c. Let the curve Abc be ſimilar to the curve ABC, and draw the right line Ag ſo as to touch both curves in A, and cut the ordinates DB, EC, db, ec, in F, G, f, g. Then, ſuppoſing the length Ae to remain the ſame, let the points B and C meet in the point A; and the angle cAg vaniſhing, the curvilinear areas Abd, Ace will coincide with the rectilinear areas Afd, Age; and therefore (by Lem. 5.) will be one to the other in the duplicate ratio of the ſides Ad, Ae. But the areas ABD, ACE are always proportional to theſe areas; and ſo the ſides AD, AE are to theſe ſides. And therefore the areas ABD, ACE are ultimately one to the other in the duplicate ratio of the ſides AD, AE. Q.E.D.

LEMMA X.

The ſpaces which a body deſcribes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminiſhed, are in the very beginning of the motion one to the other in the duplicate ratio of the times.

Let the times be repreſented by the lines AD, AE, and the velocities generated in thoſe times by the ordinates DB, EC. The ſpaces deſcribed with theſe velocities will be as the areas ABD, ACE, deſcribed by thoſe ordinates, that is, at the very beginning of the motion (by Lem. 9.), in the duplicate ratio of the times AD, AE. Q.E.D.

Cor. 1. And hence one may eaſily infer, that the errors of bodies deſcribing ſimilar parts of ſimilar figures in proportional times, are nearly in the duplicate ratio of the times in which they are generated; if ſo be theſe errors are generated by any equal forces ſimilarly applied to the bodies, and meaſured by the diſtances of the bodies from thoſe places of the

similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.

COR. 2. But the errors that are generated by proportional forces, similarly applied to the bodies at similar parts of the similar figures, are as the forces and the squares of the times conjunctly.

COR. 3. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces; all which, in the very beginning of the motion, are as the forces and the squares of the times conjunctly.

COR. 4. And therefore the forces are as the spaces described in the very beginning of the motion directly, and the squares of the times inversely.

COR. 5. And the squares of the times are as the spaces described directly, and the forces inversely.

SCHOLIUM.

If in comparing indetermined quantities of different sorts one with another, any one is said to be as any other directly or inversely, the meaning is, that the former is augmented or diminished in the same ratio with the latter, or with its reciprocal. And if any one is said to be as any other two or more directly or inversely, the meaning is, that the first is augmented or diminished in the ratio compounded of the ratios in which the others, or the reciprocals of the others, are augmented or diminished. As if A is said to be as B directly, and C directly, and D inversely, the meaning is, that A is augmented or diminished in the same ratio with $B \times C \times \frac{1}{D}$, that is to say, that A and $\frac{BC}{D}$ are one to the other in a given ratio.

LEMMA XI.

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the subtense of the conterminate arc. (Pl. 2. fig. 4.)

CASE 1. Let AB be that arc, AD its tangent, BD the subtense of the angle of contact perpendicular on the tangent, AB the subtense of the arc. Draw BG perpendicular to the subtense AB, and AG to the tangent AD, meeting in G;

then let the points D, B, and G, approach to the points d, b, and g, and suppose J to be the ultimate intersection of the lines BG, AG, when the points D, B have come to A. It is evident that the distance GJ may be less than any assignable. But (from the nature of the circles passing through the points A, B, G, A, b, g), $AB^2 = AG \times BD$, and $Ab^2 = Ag \times bd$; and therefore the ratio of AB^2 to Ab^2 is compounded of the ratios of AG to Ag, and of BD to bd. But because GJ may be assumed of less length than any assignable, the ratio of AG to Ag may be such as to differ from the ratio of equality by less than any assignable difference; and therefore the ratio of AB^2 to Ab^2 may be such as to differ from the ratio of BD to bd by less than any assignable difference. Therefore, by Lem. 1, the ultimate ratio of AB^2 to Ab^2 is the same with the ultimate ratio of BD to bd. Q.E.D.

CASE 2. Now let BD be inclined to AD in any given angle, and the ultimate ratio of BD to bd will always be the same as before, and therefore the same with the ratio of AB^2 to Ab^2. Q.E.D.

CASE 3. And if we suppose the angle D not to be given, but that the right line BD converges to a given point, or is determined by any other condition whatever; nevertheless, the angles D, d, being determined by the same law, will always draw nearer to equality, and approach nearer to each other than by any assigned difference, and therefore, by Lem. 1, will at last be equal; and therefore the lines BD, bd are in the same ratio to each other as before. Q.E.D.

COR. 1. Therefore since the tangents AD, Ad, the arcs AB, Ab, and their sines BC, bc, become ultimately equal to the chords AB, Ab, their squares will ultimately become as the subtenses BD, bd.

COR. 2. Their squares are also ultimately as the versed sines of the arcs, bisecting the chords, and converging to a given point. For those versed sines are as the subtenses BD, bd.

COR. 3. And therefore the versed sine is in the duplicate ratio of the time in which a body will describe the arc with a given velocity.

COR. 4. The rectilinear triangles ADB, Adb are ultimately in the triplicate ratio of the fides AD, Ad, and in a fefquiplicate ratio of the fides DB, db; as being in the ratio compounded of the fides AD to DB, and of Ad to db. So alfo the triangles ABC, Abc are ultimately in the triplicate ratio of the fides BC, bc. What I call the fefquiplicate ratio is the fubduplicate of the triplicate, as being compounded of the fimple and fubduplicate ratio.

COR. 5. And becaufe DB, db are ultimately parallel and in the duplicate ratio of the lines AD, Ad, the ultimate curvilinear areas ADB, Adb will be (by the nature of the parabola) two thirds of the rectilinear triangles ADB, Adb; and the fegments AB, Ab will be one third of the fame triangles. And thence thofe areas and thofe fegments will be in the triplicate ratio as well of the tangents AD, Ad, as of the chords and arcs AB, AB.

SCHOLIUM.

But we have all along fuppofed the angle of contact to be neither infinitely greater nor infinitely lefs than the angles of contact made by circles and their tangents; that is, that the curvature at the point A is neither infinitely fmall nor infinitely great, or that the interval AJ is of a finite magnitude. For DB may be taken as AD^3: in which cafe no circle can be drawn through the point A, between the tangent AD and the curve AB, and therefore the angle of contact will be infinitely lefs than thofe of circles. And by a like reafoning, if DB be made fucceffively as AD^4, AD^5, AD^6, AD^7, &c. we fhall have a feries of angles of contact, proceeding *in infinitum*, wherein every fucceeding term is infinitely lefs than the preceding. And if DB be made fucceffively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we fhall have another infinite feries of angles of contact, the firft of which is of the fame fort with thofe of circles, the fecond infinitely greater, and every fucceeding one infinitely greater than the preceding. But between any two of thefe angles another feries of intermediate angles of contact may be interpofed, proceeding both ways *in infinitum*, wherein every fucceeding angle fhall be infinitely

greater or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by infinite intervals. Nor is nature confined to any bounds.

Those things which have been demonstrated of curve lines, and the superficies which they comprehend, may be easily applied to the curve superficies and contents of solids. These lemmas are premised to avoid the tediousness of deducing perplexed demonstrations *ad absurdum*, according to the method of the antient geometers. For demonstrations are more contracted by the method of indivisibles: but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios; and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with more safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curve lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios: and that the force of such demonstrations always depends on the method laid down in the foregoing lemmas.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alledged, that a body arriving at a certain place, and there stopping, has no ultimate velocity: because the velocity, before the body comes to the place, is not its ultimate

velocity; when it has arrived, is none. But the anfwer is eafy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its laft place and the motion ceafes, nor after, but at the very inftant it arrives; that is, that velocity with which the body arrives at its laft place, and with which the motion ceafes. And in like manner, by the ultimate ratio of evanefcent quantities is to be underftood the ratio of the quantities not before they vanifh, nor afterwards, but with which they vanifh. In like manner the firft ratio of nafcent quantities is that with which they begin to be. And the firft or laft fum is that with which they begin and ceafe to be (or to be augmented or diminifhed). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and ceafe to be. And fince fuch limits are certain and definite, to determine the fame is a problem ftrictly geometrical. But whatever is geometrical we may be allowed to ufe in determining and demonftrating any other thing that is likewife geometrical.

It may alfo be objected, that if the ultimate ratios of evanefcent quantities are given, their ultimate magnitudes will be alfo given: and fo all quantities will confift of indivifibles, which is contrary to what *Euclid* has demonftrated concerning incommenfurables, in the 10th book of his Elements. But this objection is founded on a falfe fuppofition. For thofe ultimate ratios with which quantities vanifh are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreafing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminifhed *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whofe difference is given, be augmented *in infinitum,* the ultimate ratio of thefe quantities will be given, to wit, the ratio of equality; but it does not from thence follow, that the ultimate or greateft quantities themfelves, whofe ratio that is, will be given. Therefore if in what follows, for the

ſake of being more eaſily underſtood, I ſhould happen to mention quantities as leaſt, or evaneſcent, or ultimate, you are not to ſuppoſe that quantities of any determinate magnitude are meant, but ſuch as are conceived to be always diminiſhed without end.

SECTION II.

Of the Invention of Centripetal Forces.

PROPOSITION I. THEOREM I.

The areas, which revolving bodies deſcribe by radii drawn to an immovable centre of force do lie in the ſame immovable planes, and are proportional to the times in which they are deſcribed. (Pl. 2. Fig. 5.)

For ſuppoſe the time to be divided into equal parts, and in the firſt part of that time let the body by its innate force deſcribe the right line AB. In the ſecond part of that time, the ſame would (by law 1.), if not hindered, proceed directly to c, along the line Bc equal to AB; ſo that by the radii AS, BS, cS, drawn to the centre, the equal areas ABS, BSc, would be deſcribed. But when the body is arrived at B, ſuppoſe that a centripetal force acts at once with a great impulſe, and, turning aſide the body from the right line Bc, compels it afterwards to continue its motion along the right line BC. Draw cC parallel to BS meeting BC in C; and at the end of the ſecond part of the time, the body (by cor. 1. of the laws) will be found in C, in the ſame plane with the triangle ASB. Join SC, and, becauſe SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc, and therefore alſo to the triangle SAB. By the like argument, if the centripetal force acts ſucceſſively in C, D, E, &c. and makes the body, in each ſingle particle of time, to deſcribe the right lines CD, DE, EF, &c. they will all lie in the ſame plane; and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. And therefore, in equal times, equal areas are deſcribed in one immovable plane: and, by compoſition, any ſums SADS, SAFS, of thoſe areas, are one to the other as the times in which they are deſcribed. Now let the number of thoſe triangles be augmented, and their breadth diminiſhed *in infinitum;* and

(by cor. 4, lem. 3.) their ultimate perimeter ADF will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any described areas SADS, SAFS, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.

COR. 1. The velocity of a body attracted towards an immovable centre, in spaces void of resistance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A, B, C, D, E, are as the bases AB, BC, CD, DE, EF, of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them.

COR. 2. If the chords AB, BC of two arcs, successively described in equal times by the same body, in spaces void of resistance, are completed into a parallelogram ABCV, and the diagonal BV of this parallelogram, in the position which it ultimately acquires when those arcs are diminished *in infinitum*, is produced both ways, it will pass through the centre of force.

COR. 3. If the chords AB, BC, and DE, EF, of arcs described in equal times, in spaces void of resistance, are completed into the parallelograms ABCV, DEFZ; the forces in B and E are one to the other in the ultimate ratio of the diagonals BV, EZ, when those arcs are diminished *in infinitum*. For the motions BC and EF of the body (by cor. 1 of the laws) are compounded of the motions Bc, BV, and Ff, EZ: but BV and EZ, which are equal to Cc and Ff, in the demonstration of this proposition, were generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. 4. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are one to another as the versed sines of arcs described in equal times; which versed sines tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such versed sines are the halves of the diagonals mentioned in cor. 3.

Cor. 5. And therefore those forces are to the force of gravity as the said versed sines to the versed sines perpendicular to the horizon of those parabolic arcs which projectiles describe in the same time.

Cor. 6. And the same things do all hold good (by cor. 5 of the laws), when the planes in which the bodies are moved, together with the centres of force which are placed in those planes, are not at rest, but move uniformly forward in right lines.

PROPOSITION II. THEOREM II.

Every body that moves in any curve line described in a plane, and by a radius, drawn to a point either immovable, or moving forward with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

Case 1. For every body that moves in a curve line, is (by law 1) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBC, SCD, &c. about the immovable point S (by prop. 40, book 1, elem. and law 2), acts in the place B, according to the direction of a line parallel to cC, that is, in the direction of the line BS; and in the place C, according to the direction of a line parallel to dD, that is, in the direction of the line CS, &c.; and therefore acts always in the direction of lines tending to the immovable point S. Q.E.D.

Case 2. And (by cor. 5 of the laws) it is indifferent whether the superficies in which a body describes a curvilinear figure be quiescent, or moves together with the body, the figure described, and its point S, uniformly forwards in right lines.

Cor. 1. In non-resisting spaces or mediums, if the areas are not proportional to the times, the forces are not directed to the point in which the radii meet; but deviate therefrom *in consequentia*, or towards the parts to which the motion is directed, if the description of the areas is accelerated; but *in antecedentia*, if retarded.

2.

Cor. 2. And even in refifting mediums, if the defcription of the areas is accelerated, the directions of the forces deviate from the point in which the radii meet, towards the parts to which the motion tends.

SCHOLIUM.

A body may be urged by a centripetal force compounded of feveral forces; in which cafe the meaning of the propofition is, that the force which refults out of all tends to the point S. But if any force acts perpetually in the direction of lines perpendicular to the defcribed furface, this force will make the body to deviate from the plane of its motion: but will neither augment nor diminifh the quantity of the defcribed furface, and is therefore to be neglected in the compofition of forces.

PROPOSITION III. THEOREM III.

Every body, that, by a radius drawn to the centre of another body, howfoever moved, defcribes areas about that centre proportional to the times, is urged by a force compounded out of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

Let L reprefent the one, and T the other body; and (by cor. 6 of the laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the fecond body T is urged, the firft body L will go on to defcribe about the other body T the fame areas as before: but the force by which that other body T was urged will be now deftroyed by an equal and contrary force; and therefore (by law 1) that other body T, now left to itfelf, will either reft, or move uniformly forward in a right line: and the firft body L impelled by the difference of the forces, that is, by the force remaining, will go on to defcribe about the other body T areas proportional to the times. And therefore (by theor. 2) the difference of the forces is directed to the other body T as its centre. Q. E. D.

Cor. 1. Hence if the one body L, by a radius drawn to the other body T, defcribes areas proportional to the times; and from the whole force, by which the firft body L is urged

(whether that force is simple, or, according to cor. 2 of the laws, compounded out of several forces), we subduct (by the same cor.) that whole accelerative force by which the other body is urged; the whole remaining force by which the first body is urged will tend to the other body T, as its centre.

Cor. 2. And, if these areas are proportional to the times nearly, the remaining force will tend to the other body T nearly.

Cor. 3. And *vice versa,* if the remaining force tends nearly to the other body T, those areas will be nearly proportional to the times.

Cor. 4. If the body L, by a radius drawn to the other body T, describes areas, which, compared with the times, are very unequal; and that other body T be either at rest, or moves uniformly forward in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immovable or moveable) centre. The same thing obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subducting that whole force acting upon that other body T.

SCHOLIUM.

Because the equable description of areas indicates that a centre is respected by that force with which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit; why may we not be allowed, in the following discourse, to use the equable description of areas as an indication of a centre, about which all circular motion is performed in free spaces?

PROPOSITION IV. THEOREM IV.

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are one to the other as the squares of the arcs described in equal times applied to the radii of the circles.

These forces tend to the centres of the circles (by prop. 2, and cor 2, prop. 1), and are one to another as the versed sines

of the leaſt arcs deſcribed in equal times (by cor. 4, prop. 1); that is, as the ſquares of the ſame arcs applied to the diameters of the circles (by lem. 7); and therefore ſince thoſe arcs are as arcs deſcribed in any equal times, and the diameters are as the radii, the forces will be as the ſquares of any arcs deſcribed in the ſame time applied to the radii of the circles. Q.E.D.

Cor. 1. Therefore, ſince thoſe arcs are as the velocities of the bodies, the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the ſimple ratio of the radii inverſely.

Cor. 2. And ſince the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inverſely, the centripetal forces, are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodic times inverſely.

Cor. 3. Whence if the periodic times are equal, and the velocities therefore as the radii, the centripetal forces will be alſo as the radii; and the contrary.

Cor. 4. If the periodic times and the velocities are both in the ſubduplicate ratio of the radii, the centripetal forces will be equal among themſelves; and the contrary.

Cor. 5. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and the contrary.

Cor. 6. If the periodic times are in the ſeſquiplicate ratio of the radii, and therefore the velocities reciprocally in the ſubduplicate ratio of the radii, the centripetal forces will be in the duplicate ratio of the radii inverſely; and the contrary.

Cor. 7. And univerſally, if the periodic time is as any power R^n of the radius R, and therefore the velocity reciprocally as the power R^{n-1} of the radius, the centripetal force will be reciprocally as the power R^{2n-1} of the radius; and the contrary.

Cor. 8. The ſame things all hold concerning the times, the velocities, and forces by which bodies deſcribe the ſimilar parts of any ſimilar figures that have their centres in a ſimilar poſition with thoſe figures; as appears by applying the demonſtration of the preceding caſes to

thoſe. And the application is eaſy, by only ſubſtituting the equable deſcription of areas in the place of equable motion, and uſing the diſtances of the bodies from the centres inſtead of the radii.

Cor. 9. From the ſame demonſtration it likewiſe follows, that the arc which a body, uniformly revolving in a circle by means of a given centripetal force, deſcribes in any time, is a mean proportional between the diameter of the circle, and the ſpace which the ſame body falling by the ſame given force would deſcend through in the ſame given time.

SCHOLIUM.

The caſe of the 6th corollary obtains in the celeſtial bodies (as Sir *Chriſtopher Wren*, Dr. *Hooke*, and Dr. *Halley* have ſeverally obſerved); and therefore in what follows, I intend to treat more at large of thoſe things which relate to centripetal force decreaſing in a duplicate ratio of the diſtances from the centres.

Moreover, by means of the preceding propoſition and its corollaries, we may diſcover the proportion of a centripetal force to any other known force, ſuch as that of gravity. For if a body by means of its gravity revolves in a circle concentric to the earth, this gravity is the centripetal force of that body. But, from the deſcent of heavy bodies, the time of one entire revolution, as well as the arc deſcribed in any given time, is given (by cor. 9 of this prop.). And by ſuch propoſitions, Mr. *Huygens*, in is excellent book *De Horologio Oſcillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding propoſition may be likewiſe demonſtrated after this manner. In any circle ſuppoſe a polygon to be inſcribed of any number of ſides. And if a body, moved with a given velocity along the ſides of the polygon, is reflected from the circle at the ſeveral angular points, the force, with which at every reflection it ſtrikes the circle, will be as its velocity: and therefore the ſum of the forces, in a given time, will be as that velocity and the number of reflections conjunctly; that is (if the ſpecies of the polygon be given), as the length deſcribed in that given time, and increaſed or diminiſhed in

the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time applied to the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROPOSITION V. PROBLEM I.

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre. (Pl. 3. Fig. 1.)

Let the three right lines PT, TQV, VR touch the figure described in as many points P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, reciprocally proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C, of the perpendiculars draw AD, DBE, EC, at right angles, meeting in D and E: and the right lines TD, VE produced, will meet in S, the centre required.

For the perpendiculars let fall from the centre S on the tangents PT, QT, are reciprocally as the velocities of the bodies in the points P and Q (by cor. 1, prop. 1), and therefore, by construction, as the perpendiculars AP, BQ directly; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T, are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q.E.D.

PROPOSITION VI. THEOREM V.

In a space void of resistance, if a body revolves in any orbit about an immovable centre, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force in

the middle of the arc will be as the versed sine directly and the square of the time inversely.

For the versed sine in a given time is as the force (by cor. 4, prop. 1); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the versed sine will be augmented in the duplicate of that ratio (by cor. 2 and 3, lem. 11), and therefore is as the force and the square of the time. Subduct on both sides the duplicate ratio of the time, and the force will be as the versed sine directly, and the square of the time inversely. Q.E.D.

And the same thing may also be easily demonstrated by corol. 4, lem. 10.

Cor. 1. If a body P revolving about the centre S (Pl. 3, Fig. 2) describes a curve line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the solid $\frac{SP^2 \times QT^2}{QR}$, if the solid be taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the versed sine of double the arc QP, whose middle is P: and double the triangle SQP, or $SP \times QT$ is proportional to the time in which that double arc is described; and therefore may be used for the exponent of the time.

Cor. 2. By a like reasoning, the centripetal force is reciprocally as the solid $\frac{SY^2 \times QP^2}{QR}$; if SY is a perpendicular from the centre of force on PR the tangent of the orbit. For the rectangles $SY \times QP$ and $SP \times QT$ are equal.

Cor. 3. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be reciprocally as the solid $SY^2 \times PV$. For PV is $\frac{QP^2}{QR}$.

Cor. 4. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord

inversely. For the velocity is reciprocally as the perpendicular SY, by cor. 1. prop 1.

Cor. 5. Hence if any curvilinear figure APQ is given, and therein a point S is also given, to which a centripetal force is perpetually directed, that law of centripetal force may be found, by which the body P will be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, will describe the same by a perpetual revolution. That is, we are to find, by computation, either the solid $\frac{SP^2 \times QT^2}{QR}$ or the solid $SY^2 \times PV$, reciprocally proportional to this force. Examples of this we shall give in the following problems.

PROPOSITION VII. PROBLEM II.

If a body revolves in the circumference of a circle; it is proposed to find the law of centripetal force directed to any given point. (Pl. 3. Fig. 3.)

Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle at the preceding place. Through the point S draw the chord PV, and the diameter VA of the circle: join AP, and draw QT perpendicular to SP, which produced, may meet the tangent PR in Z; and lastly, through the point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, because of the similar triangles ZQR, ZTP, VPA, we shall have RP^2, that is, QRL to QT^2 as AV^2 to PV^2. And therefore $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2. Multiply those equals by $\frac{SP^2}{QR}$, and the points P and Q coinciding, for RL write PV; then we shall have $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. And therefore (by cor. 1 and 5, prop. 6) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is (because AV^2 is given), reciprocally as the square of the distance or altitude SP, and the cube of the chord PV conjunctly. Q.E.I.

The same otherwise.

On the tangent PR produced let fall the perpendicular SY; and (because of the similar triangles SYP, VPA) we shall have AV to PV as SP to SY, and therefore $\frac{SP \times PV}{AV} = SY$, and $\frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV$. And therefore (by corol. 3 and 5, prop. 6,) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is (because AV is given), reciprocally as $SP^2 \times PV^3$. Q.E.I.

COR. 1. Hence if the given point S, to which the centripetal force always tends, is placed in the circumference of the circle, as at V, the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude SP.

COR. 2. The force by which the body P in the circle APTV (Pl. 3, Fig. 4) revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \times SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction of this proposition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$; that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$; or (because of the similar triangles PSG, TPV) to SG^3.

COR. 3. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \times RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROPOSITION VIII. PROBLEM III.

If a body moves in the semi-circumference PQA; *it is proposed to find the law of the centripetal force tending to a point* S, *so remote, that all the lines* PS, RS *drawn thereto, may be taken for parallels.* (Pl. 3, Fig. 5.)

From C, the centre of the semi-circle, let the semi-diameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM, PZT, and RZQ, we shall have CP^2 to PM^2 as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or, the points P, Q coinciding, to the rectangle $QR \times 2PM$. Therefore CP^2 is to PM^2 as $QR \times 2PM$ to QT^2; and $\frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$. And therefore (by corol. 1 and 5, prop. 6) the centripetal force is reciprocally as $\frac{2PM^3 \times SP^2}{CP^2}$; that is (neglecting the given ratio $\frac{2SP^2}{CP^2}$), reciprocally as PM^3. Q.E.I.

And the same thing is likewise easily inferred from the preceding proposition.

SCHOLIUM.

And by a like reasoning, a body will be moved in an ellipsis, or even in an hyperbola, or parabola, by a centripetal force which is reciprocally as the cube of the ordinate directed to an infinitely remote centre of force.

PROPOSITION IX. PROBLEM IV.

If a body revolves in a spiral PQS, *cutting all the radii* SP, SQ, *&c. in a given angle; it is proposed to find the law of the centripetal force tending to the centre of that spiral.* (Pl. 3, Fig. 6.)

Suppose the indefinitely small angle PSQ to be given; because, then, all the angles are given, the figure SPRQT will be given in specie. Therefore the ratio $\frac{QT}{QR}$ is also given, and $\frac{QT^2}{QR}$ is as QT, that is (because the figure is given in specie), as

SP. But if the angle PSQ is any way changed, the right line QR, ſubtending the angle of contact QPR (by lem. 11) will be changed in the duplicate ratio of PR or QT. Therefore the ratio $\frac{QT^2}{QR}$ remains the ſame as before, that is, as SP. And $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and therefore (by corol. 1 and 5, prop. 6) the centripetal force is reciprocally as the cube of the diſtance SP. Q.E.I.

The ſame otherwiſe.

The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the ſpiral, are in given ratios to the height SP; and therefore SP^3 is as $SY^2 \times PV$, that is (by corol. 3 and 5, prop. 6), reciprocally as the centripetal force.

LEMMA XII.

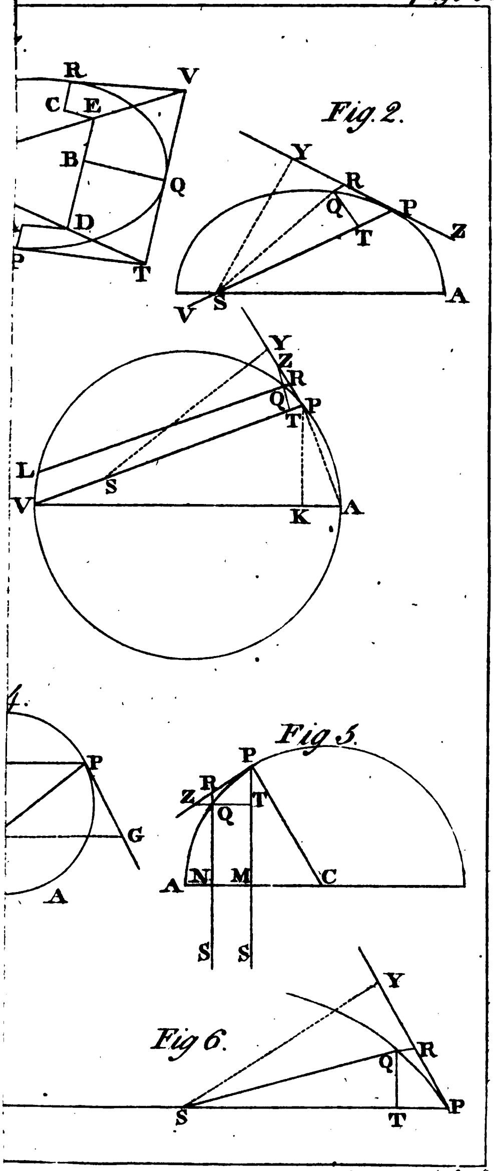
All parallelograms circumſcribed about any conjugate diameters of a given ellipſis or hyperbola are equal among themſelves.

This is demonſtrated by the writers on the conic ſections.

PROPOSITION X. PROBLEM V.

If a body revolves in an ellipſis; it is propoſed to find the law of the centripetal force tending to the centre of the ellipſis. (Pl. 4, Fig. 1.)

Suppoſe CA, CB to be ſemi-axes of the ellipſis GP, DK, conjugate diameters; PF, QT perpendiculars to thoſe diameters; Qv an ordinate to the diameter GP; and if the parallelogram QvPR be completed, then (by the properties of the conic ſections) the rectangle PvG will be to Qv^2 as PC^2 to CD^2; and (becauſe of the ſimilar triangles QvT, PCF) Qv^2 to QT^2 as PC^2 to PF^2; and, by compoſition, the ratio of PvG to QT^2 is compounded of the ratio of PC^2 to CD^2, and of the ratio of PC^2 to PF^2, that is, vG to $\frac{QT^2}{Pv}$ as PC^2 to $\frac{CD^2 \times PF^2}{PC^2}$. Put QR for Pv, and (by lem. 12) $BC \times CA$ for $CD \times PF$; alſo (the points P and Q coinciding) 2PC for vG; and multiplying the extremes and means together,



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we shall have $\frac{QT^2 \times PC^2}{QR}$ equal to $\frac{2BC^2 \times CA^2}{PC}$. Therefore (by cor. 5, prop. 6) the centripetal force is reciprocally as $\frac{2BC^2 \times CA^2}{PC}$; that is (because $2BC^2 \times CA^2$ is given), reciprocally as $\frac{1}{PC}$; that is, directly as the distance PC. Q.E.I.

The same otherwise.

In the right line PG on the other side of the point T, take the point u so that Tu may be equal to Tv; then take nV, such as shall be to vG as DC^2 to PC^2. And because Qv^2 is to PvG as DC^2 to PC^2 (by the conic sections), we shall have $Qv^2 = Pv \times uV$. Add the rectangle uPv to both sides, and the square of the chord of the arc PQ will be equal to the rectangle VPv; and therefore a circle which touches the conic section in P, and passes through the point Q, will pass also through the points V. Now let the points P and Q meet, and the ratio of uV to vG, which is the same with the ratio of DC^2 to PC^2, will become the ratio of PV to PG, or PV to 2PC; and therefore PV will be equal to $\frac{2DC^2}{PC}$. And therefore the force by which the body P revolves in the ellipsis will be reciprocally as $\frac{2DC^2}{PC} \times PF^2$ (by cor. 3, prop. 6); that is (because $2DC^2 \times PF^2$ is given), directly as PC. Q.E.I.

Cor. 1. And therefore the force is as the distance of the body from the centre of the ellipsis; and, *vice versa*, if the force is as the distance, the body will move in an ellipsis whose centre coincides with the centre of force, or perhaps in a circle into which the ellipsis may degenerate.

Cor. 2. And the periodic times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar ellipses will be equal (by corol. 3 and 8, prop. 4); but in ellipses that have their greater axis common, they are one to another as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the lesser axes directly, and the velocities of the bodies in their principal vertices inversely; that is, as those lesser axes directly, and the

ordinates to the ſame point of the common axis inverſely; and therefore (becauſe of the equality of the direct and inverſe ratios) in the ratio of equality.

SCHOLIUM.

If the ellipſis, by having its centre removed to an infinite diſtance, degenerates into a parabola, the body will move in this parabola; and the force, now tending to a centre infinitely remote, will become equable. Which is *Galileo's* theorem. And if the parabolic ſection of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipſis, if the forces are directed to the centre of the figure placed in the abſciſſa, thoſe forces, by increaſing or diminiſhing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abſciſſa, are always augmented or diminiſhed in the ratio of the diſtances from the centre; provided the periodic times remain equal: ſo alſo in all figures whatſoever, if the ordinates are augmented or diminiſhed in any given ratio, or their inclination is any way changed, the periodic time remaining the ſame, the forces directed to any centre placed in the abſciſſa are in the ſeveral ordinates augmented or diminiſhed in the ratio of the diſtances from the centre.

SECTION III.

Of the motion of bodies in eccentric conic ſections.

PROPOSITION XI. PROBLEM VI.

If a body revolves in an ellipſis; it is required to find the law of the centripetal force tending to the focus of the ellipſis. (Pl. 4, Fig. 2.)

Let S be the focus of the ellipſis. Draw SP cutting the diameter DK of the ellipſis in E, and the ordinate Qv in x; and complete the parallelogram QxPR. It is evident that EP is equal to the greater ſemi-axis AC: for drawing HI from the other focus H of the ellipſis parallel to EC, becauſe CS, CH are equal ES, EI will be alſo equal; ſo that EP is the

half ſum of PS, PI, that is (becauſe of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipſis (or for $\frac{2BC^2}{AC}$), we ſhall have $L \times QR$ to $L \times Pv$ as QR to Pv, that is, as PE or AC to PC; and $L \times Pv$ to GvP as L to Gv; and GvP to Qv^2 as PC^2 to CD^2; and (by corol. 2, lem. 7) the points Q and P coinciding, Qv^2 is to Qx^2 in the ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by lem. 12) as CD^2 to CB^2. And compounding all thoſe ratios together, we ſhall have $L \times QR$ to QT^2 as $AC \times L \times PC^2 \times CD^2$, or $2CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as 2PC to Gv. But the points Q and P coinciding, 2PC and Gv are equal. And therefore the quantities $L \times QR$ and QT^2, proportional to theſe, will be alſo equal. Let thoſe equals be drawn into $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by corol. 1 and 5, prop. 6) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ratio of the diſtance SP. Q.E.I.

The ſame otherwiſe.

Seeing the force tending to the centre of the ellipſis, by which the body P may revolve in that ellipſis, is (by corol. 1. prop. 10) as the diſtance CP of the body from the centre C of the ellipſis; let CE be drawn parallel to the tangent PR of the ellipſis; and the force by which the ſame body P may revolve about any other point S of the ellipſis, if CE and PS interſect in E, will be as $\frac{PE^3}{SP^2}$ (by cor. 3, prop. 7); that is, if the point S is the focus of the ellipſis, and therefore PE be given, as SP^2 reciprocally. Q.E.I.

With the ſame brevity with which we reduced the fifth problem to the parabola and hyperbola, we might do the like here: but becauſe of the dignity of the problem and its uſe in what follows, I ſhall confirm the other caſes by particular demonſtrations.

PROPOSITION XII. PROBLEM VII.

Suppose a body to move in an hyperbola; it is required to find the law of the centripetal force tending to the focus of that figure. (Pl. 5, Fig. 1.)

Let CA, CB be the semi-axes of the hyperbola; PG, KD other conjugate diameters; PF a perpendicular to the diameter KD; and Qv an ordinate to the diameter GP. Draw SP cutting the diameter DK in E, and the ordinate Qv in x, and complete the parallelogram QRPx. It is evident that EP is equal to the semi-transverse axis AC; for drawing HI, from the other focus H of the hyperbola, parallel to EC; because CS, CH are equal, ES, EI will be also equal; so that EP is the half difference of PS, PI; that is (because of the parallels IH, PR, and the equal angles IPR, HPZ), of PS, PH, the difference of which is equal to the whole axis 2AC. Draw QT perpendicular to SP; and putting L for the principal latus rectum of the hyperbola (that is, for $\frac{2BC^2}{AC}$), we shall have $L \times QR$ to $L \times Pv$ as QR to Pv, or Px to Pv, that is (because of the similar triangles Pxv, PEC), as PE to PC, or AC to PC. And $L \times Pv$ will be to $Gv \times Pv$ as L to Gv; and (by the properties of the conic sections) the rectangle GvP is to Qv^2 as PC^2 to CD^2; and (by cor. 2, lem. 7), Qv^2 to Qx^2, the points Q and P coinciding, becomes a ratio of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by lem. 12) as CD^2 to CB^2: and, compounding all those ratios together, we shall have $L \times QR$ to QT^2 as $AC \times L \times PC^2 \times CD^2$, or $2CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as 2PC to Gv. But the points P and Q coinciding, 2PC and Gv are equal. And therefore the quantities $L \times QR$ and QT^2, proportional to them, will be also equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and we shall have $L \times SP^2$ equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by cor. 1 & 5, prop. 6) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ratio of the distance SP. Q.E.I.

Fig 1.

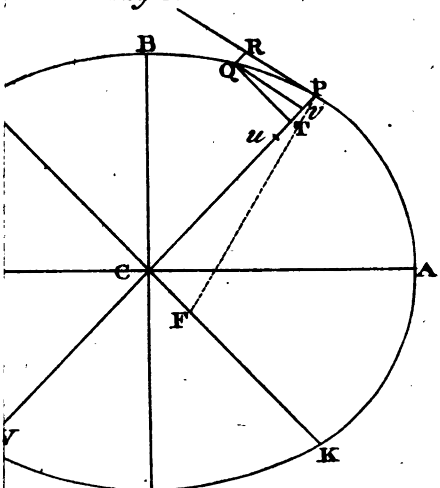
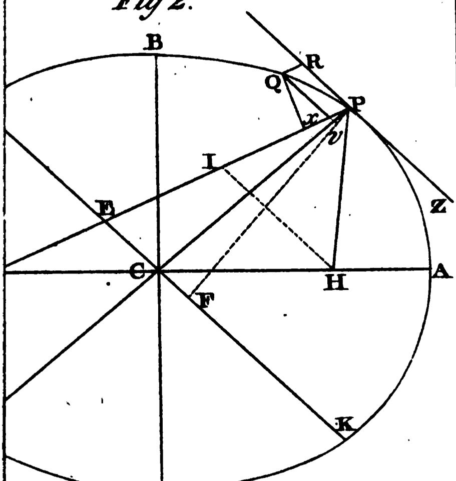


Fig 2.



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The same otherwise.

Find out the force tending from the centre C of the hyperbola. This will be proportional to the distance CP. But from thence (by cor. 3, prop. 7) the force tending to the focus S will be as $\frac{PE^3}{SP^2}$, that is, because PE is given, reciprocally as SP^2. Q.E.I.

And the same way it may be demonstrated, that the body having its centripetal changed into a centrifugal force, will move in the conjugate hyperbola.

LEMMA XIII.

The latus rectum of a parabola belonging to any vertex is quadruple the distance of that vertex from the focus of the figure.

This is demonstrated by the writers on the conic sections.

LEMMA XIV.

The perpendicular let fall from the focus of a parabola on its tangent, is a mean proportional between the distances of the focus from the point of contact, and from the principal vertex of the figure. (Pl. 5, Fig. 2.)

For, let AP be the parabola, S its focus, A its principal vertex, P the point of contact, PO an ordinate to the principal diameter, PM the tangent meeting the principal diameter in M, and SN the perpendicular from the focus on the tangent: join AN, and because of the equal lines MS and SP, MN and NP, MA and AO, the right lines AN, OP, will be parallel; and thence the triangle SAN will be right angled at A, and similar to the equal triangles SNM, SNP; therefore PS is to SN as SN to SA. Q.E.D.

Cor. 1. PS^2 is to SN^2 as PS to SA.

Cor. 2. And because SA is given, SN^2 will be as PS.

Cor. 3. And the concourse of any tangent PM, with the right line SN, drawn from the focus perpendicular on the tangent, falls in the right line AN, that touches the parabola in the principal vertex.

PROPOSITION XIII., PROBLEM VIII.

If a body moves in the perimeter of a parabola; it is required to find the law of the centripetal force tending to the focus of that figure. (Pl. 5, Fig. 3.)

Retaining the construction of the preceding lemma, let P be the body in the perimeter of the parabola; and from the place Q, into which it is next to succeed, draw QR parallel and QT perpendicular to SP, as also Qv parallel to the tangent, and meeting the diameter PG in v, and the distance SP in x. Now, because of the similar triangles Pxv, SPM, and of the equal sides SP, SM of the one, the sides Px or QR and Pv of the other, will be also equal. But (by the conic sections) the square of the ordinate Qv is equal to the rectangle under the latus rectum and the segment Pv of the diameter; that is (by lem. 13), to the rectangle $4PS \times Pv$, or $4PS \times QR$; and the points P and Q coinciding, the ratio of Qv to Qx (by cor. 2, lem. 7) becomes a ratio of equality. And therefore Qx^2, in this case, becomes equal to the rectangle $4PS \times QR$. But (because of the similar triangles QxT, SPN) Qx^2 is to QT^2 as PS^2 to SN^2, that is (by cor. 1, lem. 14), as PS to SA; that is, as $4PS \times QR$ to $4SA \times QR$, and therefore (by prop. 9, lib. 5, elem.) QT^2 and $4SA \times QR$ are equal. Multiply these equals by $\frac{SP^2}{QR}$, and $\frac{SP^2 \times QT^2}{QR}$ will become equal to $SP^2 \times 4SA$: and therefore (by cor. 1 and 5, prop. 6.), the centripetal force is reciprocally as $SP^2 \times 4SA$; that is, because 4SA is given, reciprocally in the duplicate ratio of the distance SP. Q E.I.

COR. 1. From the three last propositions it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR, and at the same time is urged by the action of a centripetal force that is reciprocally proportional to the square of the distance of the places from the centre, the body will move in one of the conic sections, having its focus in the centre of force; and the contrary. For the focus, the point of contact, and the position of the tangent, being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force, and the bodies' velocity given; and two orbits, mutually touching one the other, cannot be described by the same centripetal force and the same velocity.

8.

Cor. 2. If the velocity with which the body goes from its place P is ſuch, that in any infinitely ſmall moment of time the lineolæ PR may be thereby deſcribed; and the centripetal force ſuch as in the ſame time to move that body through the ſpace QR; the body will move in one of the conic ſections, whoſe principal latus rectum is the quantity $\frac{QT^2}{QR}$ in its ultimate ſtate, when the lineolæ PR, QR are diminiſhed *in infinitum*. In theſe corollaries I conſider the circle as an ellipſis; and I except the caſe where the body deſcends to the centre in a right line.

PROPOSITION XIV. THEOREM VI.

If ſeveral bodies revolve about one common centre, and the centripetal force is reciprocally in the duplicate ratio of the diſtance of places from the centre; I ſay, that the principal latera recta of their orbits are in the duplicate ratio of the areas, which the bodies by radii drawn to the centre deſcribe in the ſame time. (Pl. 6, Fig. 1.)

For (by cor. 2, prop. 13) the latus rectum L is equal to the quantity $\frac{QT^2}{QR}$ in its ultimate ſtate when the points P and Q coincide. But the lineolæ QR in a given time is as the generating centripetal force; that is (by ſuppoſition), reciprocally as SP^2. And therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$; that is, the latus rectum L is in the duplicate ratio of the area $QT \times SP$. Q.E.D.

Cor. Hence the whole area of the ellipſis, and the rectangle under the axes, which is proportional to it, is in the ratio compounded of the ſubduplicate ratio of the latus rectum, and the ratio of the periodic time. For the whole area is as the area $QT \times SP$, deſcribed in a given time, multiplied by the periodic time.

PROPOSITION XV. THEOREM VII.

The ſame things being ſuppoſed, I ſay, that the periodic times in ellipſes are in the ſeſquiplicate ratio of their greater axes.

For the leſſer axis is a mean proportional between the greater axis and the latus rectum; and, therefore, the rectangle under the axes is in the ratio compounded of the ſubduplicate ratio of the latus rectum and the ſeſquiplicate ratio of the greater axis. But this rectangle (by cor. 3, prop. 14) is in a ratio compounded of the ſubduplicate ratio of the latus rectum, and the ratio of the periodic time. Subduct from both ſides the ſubduplicate ratio of the latus rectum, and there will remain the ſeſquiplicate ratio of the greater axis, equal to the ratio of the periodic time. Q.E.D.

Cor. Therefore the periodic times in ellipſes are the ſame as in circles whoſe diameters are equal to the greater axes of the ellipſes.

PROPOSITION XVI. THEOREM VIII.

The ſame things being ſuppoſed, and right lines being drawn to the bodies that ſhall touch the orbits, and perpendiculars being let fall on thoſe tangents from the common focus; I ſay, that the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inverſely, and the ſubduplicate ratio of the principal latera recta directly. (Pl. 6. Fig. 2.)

From the focus S draw SY perpendicular to the tangent PR, and the velocity of the body P will be reciprocally in the ſubduplicate ratio of the quantity $\frac{SY^2}{L}$. For that velocity is as the infinitely ſmall arc PQ deſcribed in a given moment of time, that is (by lem. 7), as the tangent PR; that is, (becauſe of the proportionals PR to QT, and SP to SY), as $\frac{SP \times QT}{SY}$; or as SY reciprocally, and SP × QT directly; but SP × QT is as the area deſcribed in the given time, that is (by prop. 14), in the ſubduplicate ratio of the latus rectum. Q.E.D.

Cor. 1. The principal latera recta are in a ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

Cor. 2. The velocities of bodies, in their greateſt and leaſt diſtances from the common focus, are in the ratio compound-

ed of the ratio of the distances inversely, and the subduplicate ratio of the principal latera recta directly. For those perpendiculars are now the distances.

COR. 3. And therefore the velocity in a conic section, at its greatest or least distance from the focus, is to the velocity in a circle, at the same distance from the centre, in the subduplicate ratio of the principal latus rectum to the double of that distance.

COR. 4. The velocities of the bodies revolving in ellipses, at their mean distances from the common focus, are the same as those of bodies revolving in circles, at the same distances; that is (by cor. 6, prop. 4), reciprocally in the subduplicate ratio of the distances. For the perpendiculars are now the lesser semi-axes, and these are as mean proportionals between the distances and the latera recta. Let this ratio inversely be compounded with the subduplicate ratio of the latera recta directly, and we shall have the subduplicate ratio of the distance inversely.

COR. 5. In the same figure, or even in different figures, whose principal latera recta are equal, the velocity of a body is reciprocally as the perpendicular let fall from the focus on the tangent.

COR. 6. In a parabola, the velocity is reciprocally in the subduplicate ratio of the distance of the body from the focus of the figure; it is more variable in the ellipsis, and less in the hyperbola, than according to this ratio. For (by cor. 2, lem. 14) the perpendicular let fall from the focus on the tangent of a parabola is in the subduplicate ratio of the distance. In the hyperbola the perpendicular is less variable; in the ellipsis more.

COR. 7. In a parabola, the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle, at the same distance from the centre, in the subduplicate ratio of the number 2 to 1; in the ellipsis it is less, and in the hyperbola greater, than according to this ratio. For (by cor. 2 of this prop.) the velocity at the vertex of a parabola is in this ratio, and (by cor. 6. of this prop. and prop. 4) the same proportion holds in all distances. And hence,

alſo, in a parabola, the velocity is every where equal to the velocity of a body revolving in a circle at half the diſtance; in the ellipſis it is leſs, and in the hyperbola greater.

COR. 8. The velocity of a body revolving in any conic ſection is to the velocity of a body revolving in a circle, at the diſtance of half the principal latus rectum of the ſection, as that diſtance to the perpendicular let fall from the focus on the tangent of the ſection. This appears from cor. 5.

COR. 9. Wherefore ſince (by cor. 6, prop. 4.) the velocity of a body revolving in this circle is to the velocity of another body revolving in any other circle reciprocally in the ſubduplicate ratio of the diſtances; therefore, *ex æquo*, the velocity of a body revolving in a conic ſection will be to the velocity of a body revolving in a circle at the ſame diſtance as a mean proportional between that common diſtance, and half the principal latus rectum of the ſection, to the perpendicular let fall from the common focus upon the tangent of the ſection.

PROPOSITION XVII. PROBLEM IX.

Suppoſing the centripetal force to be reciprocally proportional to the ſquares of the diſtances of places from the centre, and that the abſolute quantity of that force is known; it is required to determine the line which a body will deſcribe that is let go from a given place with a given velocity in the direction of a given right line.

Let the centripetal force tending to the point S (Pl. 6, Fig. 3) be ſuch, as will make the body p revolve in any given orbit pq; and ſuppoſe the velocity of this body in the place p is known. Then from the place P ſuppoſe the body P to be let go with a given velocity in the direction of the line PR; but by virtue of a centripetal force to be immediately turned aſide from that right line into the conic ſection PQ. This, the right line PR will therefore touch in P. Suppoſe likewiſe that the right line pr touches the orbit pq in p; and if from S you ſuppoſe perpendiculars let fall on thoſe tangents, the principal latus rectum of the conic ſection (by cor. 1, prop. 16) will be to the principal latus rectum of that orbit in a ratio compounded of the duplicate ratio of the perpendiculars, and the dupli-

cate ratio of the velocities; and is therefore given. Let this latus rectum be L: the focus S of the conic section is also given. Let the angle RPH be the complement of the angle RPS to two right; and the line PH, in which the other focus H is placed, is given by position. Let fall SK perpendicular on PH, and erect the conjugate semi-axis BC; this done, we shall have $SP^2 - 2KPH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 = \overline{SP + PH}^2 - L \times \overline{SP + PH} = SP^2 + 2SPH + PH^2 - L \times \overline{SP + PH}$. Add on both sides $2KPH - SP^2 - PH^2 + L \times \overline{SP + PH}$, and we shall have $L \times \overline{SP + PH} = 2SPH + 2KPH$, or SP + PH to PH, as 2SP + 2KP to L. Whence PH is given both in length and position. That is, if the velocity of the body in P is such that the latus rectum L is less than 2SP + 2KP, PH will lie on the same side of the tangent PR with the line SP; and therefore the figure will be an ellipsis, which from the given foci S, H, and the principal axis SP + PH, is given also. But if the velocity of the body is so great, that the latus rectum L becomes equal to 2PS + 2KP, the length PH will be infinite; and therefore, the figure will be a parabola, which has its axis SH parallel to the line PK, and is thence given. But if the body goes from its place P with a yet greater velocity, the length PH is to be taken on the other side the tangent; and so the tangent passing between the foci, the figure will be an hyperbola having its principal axis equal to the difference of the lines SP and PH, and thence is given. For if the body, in these cases, revolves in a conic section so found, it is demonstrated in prop. 11, 12, and 13, that the centripetal force will be reciprocally as the square of the distance of the body from the centre of force S; and therefore we have rightly determined the line PQ, which a body let go from a given place P with a given velocity, and in the direction of the right line PR given by position, would describe with such a force. Q.E.F.

Cor. 1. Hence in every conic section, from the principal vertex D, the latus rectum L, and the focus S given, the other focus H is given, by taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS. For the proportion, SP + PH to PH as 2PS + 2KP to L, becomes,

in the case of this corollary, DS + DH to DH as 4DS to L, and by division DS to DH as 4DS — L to L.

COR. 2. Whence if the velocity of a body in the principal vertex D is given, the orbit may be readily found; to wit, by taking its latus rectum to twice the distance DS, in the duplicate ratio of this given velocity to the velocity of a body revolving in a circle at the distance DS (by cor. 3, prop. 16), and then taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS.

COR. 3. Hence also if a body move in any conic section, and is forced out of its orbit by any impulse, you may discover the orbit in which it will afterwards pursue its course. For by compounding the proper motion of the body with that motion, which the impulse alone would generate, you will have the motion with which the body will go off from a given place of impulse in the direction of a right line given in position.

COR. 4. And if that body is continually disturbed by the action of some foreign force, we may nearly know its course, by collecting the changes which that force introduces in some points, and estimating the continual changes it will undergo in the intermediate places, from the analogy that appears in the progress of the series.

SCHOLIUM.

If a body P (Pl. 6, Fig. 4), by means of a centripetal force tending to any given point R, move in the perimeter of any given conic section whose centre is C; and the law of the centripetal force is required: draw CG parallel to the radius RP, and meeting the tangent PG of the orbit in G; and the force required (by cor. 1, and schol. prop. 10, and cor. 3, prop. 7) will be as $\frac{CG^3}{RP^2}$.

SECTION IV.

Of the finding of elliptic, parabolic, and hyperbolic orbits, from the focus given.

LEMMA XV.

If from the two foci S, H (Pl. 7, Fig. 1), *of any ellipsis or hyperbola, we draw to any third point* V *the right lines* SV, HV, *whereof one* HV *is equal to the principal axis of the*

figure, that is, to the axis in which the foci are situated, the other, SV, *is bisected in* T *by the perpendicular* TR *let fall upon it; that perpendicular* TR *will somewhere touch the conic section: and,* vice versa, *if it does touch it,* HV *will be equal to the principal axis of the figure.*

For, let the perpendicular TR cut the right line HV, produced, if need be, in R; and join SR. Because TS, TV are equal, therefore the right lines SR, VR, as well as the angles TRS, TRV, will be also equal. Whence the point R will be in the conic section, and the perpendicular TR will touch the same; and the contrary. Q.E.D.

PROPOSITION XVIII. PROBLEM X.

From a focus and the principal axes given, to describe elliptic and hyperbolic trajectories, which shall pass through given points, and touch right lines given by position. (Pl. 7, Fig. 2.)

Let S be the common focus of the figures; AB the length of the principal axis of any trajectory; P a point through which the trajectory should pass; and TR a right line which it should touch. About the centre P, with the interval AB — SP, if the orbit is an ellipsis, or AB + SP, if the orbit is an hyperbola, describe the circle HG. On the tangent TR let fall the perpendicular ST, and produce the same to V, so that TV may be equal to ST; and about V as a centre with the interval AB describe the circle FH. In this manner, whether two points P, p, are given, or two tangents TR, tr, or a point P and a tangent TR, we are to describe two circles. Let H be their common intersection, and from the foci S, H, with the given axis describe the trajectory: I say, the thing is done. For (because PH + SP in the ellipsis, and PH — SP in the hyperbola, is equal to the axis) the described trajectory will pass through the point P, and (by the preceding lemma) will touch the right line TR. And by the same argument it will either pass through the two points P, p, or touch the two right lines TR, tr. Q.E.F.

PROPOSITION XIX. PROBLEM XI.

About a given focus, to describe a parabolic trajectory, which shall pass through given points, and touch right lines given by position. (Pl. 7, Fig. 3.)

Let S be the focus, P a point, and TR a tangent of the trajectory to be defcribed. About P as a centre, with the interval PS, defcribe the circle FG. From the focus let fall ST perpendicular on the tangent, and produce the fame to V, fo as TV may be equal to ST. After the fame manner another circle fg is to be defcribed, if another point p is given; or another point v is to be found, if another tangent tr is given; then draw the right line IF, which fhall touch the two circles FG, fg, if two points P, p are given; or pafs through the two points V, v, if two tangents TR, tr are given: or touch the circle FG, and pafs through the point V, if the point P and the tangent TR are given. On FI let fall the perpendicular SI, and bifect the fame in K; and with the axis SK and principal vertex K defcribe a parabola: I fay the thing is done. For this parabola (becaufe SK is equal to IK, and SP to FP) will pafs through the point P; and (by cor. 3, lem 14) becaufe ST is equal to TV, and STR a right angle, it will touch the right line TR. Q.E.F.

PROPOSITION XX. PROBLEM XII.

About a given focus to defcribe any trajectory given in fpecie which fhall pafs through given points, and touch right lines given by pofition.

Case 1. About the focus S (Pl. 7, Fig. 4) it is required to defcribe a trajectory ABC, paffing through two points B, C. Becaufe the trajectory is given in fpecie, the ratio of the principal axis to the diftance of the foci will be given. In that ratio take KB to BS, and LC to CS. About the centres B, C, with the intervals BK, CL, defcribe two circles; and on the right line KL, that touches the fame in K and L, let fall the perpendicular SG; which cut in A and a, fo that GA may be to AS, and Ga to aS, as KB to BS; and with the axis Aa, and vertices A, a, defcribe a trajectory: I fay the the thing is done. For let H be the other focus of the defcribed figure, and feeing GA is to AS as Ga to aS, then by divifion we fhall have Ga—GA, or Aa to aS—AS, or SH in the fame ratio, and therefore in the ratio which the principal axis of the figure to be defcribed has to the diftance of its foci; and therefore the defcribed figure is of the fame fpecies with the figure

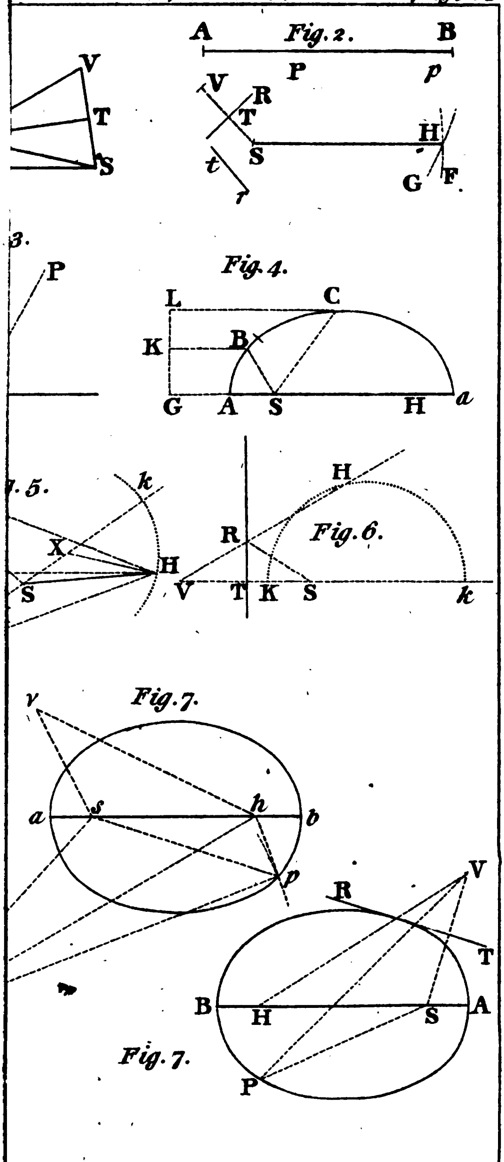
which was to be defcribed. And fince KB to BS, and LC to CS, are in the fame ratio, this figure will pafs through the points B, C, as is manifeft from the conic fections.

CASE 2. About the focus S (Pl. 7, Fig. 5) it is required to defcribe a trajectory which fhall fomewhere touch two right lines TR, tr. From the focus on thofe tangents let fall the perpendiculars ST, St, which produce to V, v, fo that TV, tv may be equal to TS, tS. Bifect Vv in O, and erect the indefinite perpendicular OH, and cut the right line VS infinitely produced in K and k, fo that VK be to KS, and Vk to kS, as the principal axis of the trajectory to be defcribed is to the diftance of its foci. On the diameter Kk defcribe a circle cutting OH in H; and with the foci S, H, and principal axis equal to VH, defcribe a trajectory: I fay, the thing is done. For, bifecting Kk in X, and joining HX, HS, HV, Hv, becaufe VK is to KS as Vk to kS; and by compofition, as VK + Vk to KS + kS; and by divifion, as Vk — VK to kS — KS, that is, as 2VX to 2KX, and 2KX to 2SX, and therefore as VX to HX and HX to SX, the triangles VXH, HXS will be fimilar; therefore VH will be to SH as VX to XH; and therefore as VK to KS. Wherefore VH, the principal axis of the defcribed trajectory, has the fame ratio to SH, the diftance of the foci, as the principal axis of the trajectory which was to be defcribed has to the diftance of its foci; and is therefore of the fame fpecies. And feeing VH, vH, are equal to the principal axis, and VS, vS are perpendicularly bifected by the right lines TR, tr, it is evident (by lem. 15) that thofe right lines touch the defcribed trajectory. Q.E.F.

CASE 3. About the focus S (Pl. 7, Fig. 6) it is required to defcribe a trajectory, which fhall touch a right line TR in a given point R. On the right line TR let fall the perpendicular ST, which produce to V, fo that TV may be equal to ST; join VR, and cut the right line VS indefinitely produced in K and k, fo that VK may be to SK, and Vk to Sk, as the principal axis of the ellipfis to be defcribed to the diftance of its foci; and on the diameter Kk defcribing a circle, cut the right line VR produced in H; then with the foci S, H,

and principal axis equal to VH, defcribe a trajectory: I fay, the thing is done. For VH is to SH as VK to SK, and therefore as the principal axis of the trajectory which was to be defcribed to the diftance of its foci (as appears from what we have demonftrated in cafe 2); and therefore the defcribed trajectory is of the fame fpecies with that which was to be defcribed; but that the right line TR, by which the angle VRS is bifected, touches the trajectory in the point R, is certain from the properties of the conic fections. Q E.F.

CASE 4. About the focus S (Plate 7, Fig. 7) it is required to defcribe a trajectory APB that fhall touch a right line TR, and pafs through any given point P without the tangent, and fhall be fimilar to the figure apb, defcribed with the principal axis ab, and foci s, h. On the tangent TR let fall the perpendicular ST, which produce to V, fo that TV may be equal to ST; and, making the angles hsq, shq equal to the angles VSP, SVP, about q as a centre, and with an interval which fhall be to ab as SP to VS, defcribe a circle cutting the figure apb in p: join sp, and draw SH fuch that it may be to sh as SP is to sp, and may make the angle PSH equal to the angle psh, and the angle VSH equal to the angle psq. Then with the foci S, H, and principal axis AB, equal to the diftance VH, defcribe a conic fection: I fay, the thing is done; for if sv is drawn fo that it fhall be to sp as sh is to sq, and fhall make the angle vsp equal to the angle hsq, and the angle vsh equal to the angle psq, the triangles svh, spq, will be fimilar, and therefore vh will be to pq as sh is to sq; that is (becaufe of the fimilar triangles VSP, hsq), as VS is to SP, or as ab to pq. Wherefore vh and ab are equal. But, becaufe of the fimilar triangles VSH, vsh, VH is to SH as vh to sh; that is, the axis of the conic fection now defcribed is to the diftance of its foci as the axis ab to the diftance of the foci sh; and therefore the figure now defcribed is fimilar to the figure apb. But, becaufe the triangle PSH is fimilar to the triangle psh, this figure paffes through the point P; and becaufe VH is equal to its axis, and VS is perpendicularly bifected by the right line TR, the faid figure touches the right line TR. Q.E.F.



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LEMMA XVI.

From three given points to draw to a fourth point that is not given three right lines whose differences shall be either given, or none at all.

CASE 1. Let the given points be A, B, C (Pl. 8, Fig. 1), and Z the fourth point which we are to find; because of the given difference of the lines AZ, BZ, the locus of the point Z will be an hyperbola whose foci are A and B, and whose principal axis is the given difference. Let that axis be MN. Taking PM to MA as MN is to AB, erect PR perpendicular to AB, and let fall ZR perpendicular to PR; then, from the nature of the hyperbola, ZR will be to AZ as MN is to AB. And by the like argument, the locus of the point Z will be another hyperbola, whose foci are A, C, and whose principal axis is the difference between AZ and CZ; and QS a perpendicular on AC may be drawn, to which (QS) if from any point Z of this hyperbola a perpendicular ZS is let fall, this (ZS) shall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratios of ZR and ZS to AZ are given, and consequently the ratio of ZR to ZS one to the other; and therefore if the right lines RP, SQ, meet in T, and TZ and TA are drawn, the figure TRZS will be given in specie, and the right line TZ, in which the point Z is somewhere placed, will be given in position. There will be given also the right line TA, and the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other is given also; and thence will be given likewise the triangle ATZ, whose vertex is the point Z. Q.E.I.

CASE 2. If two of the three lines, for example AZ and BZ, are equal, draw the right line TZ so as to bisect the right line AB; then find the triangle ATZ as above. Q.E.I.

CASE 3. If all the three are equal, the point Z will be placed in the centre of a circle that passes through the points A, B, C. Q.E.I.

This problematic lemma is likewise solved in *Apollonius's* Book of Tactions restored by *Vieta*.

PROPOSITION XXI. PROBLEM XIII.

About a given focus to describe a trajectory that shall pass through given points and touch right lines given by position.

Let the focus S (Pl. 8, Fig. 2), the point P, and the tan-

gent TR be given, and suppose that the other focus H is to be found. On the tangent let fall the perpendicular ST, which produce to Y, so that TY may be equal to ST, and YH will be equal to the principal axis. Join SP, HP, and SP will be the difference between HP and the principal axis. After this manner, if more tangents TR are given, or more points P, we shall always determine as many lines YH, or PH, drawn from the said points Y or P, to the focus H, which either shall be equal to the axes, or differ from the axes by given lengths SP; and therefore which shall either be equal among themselves, or shall have given differences; from whence (by the preceding lemma) that other focus H is given. But having the foci and the length of the axis (which is either YH, or, if the trajectory be an ellipsis, PH + SP; or PH — SP, if it be an hyperbola), the trajectory is given. Q.E.I.

SCHOLIUM.

When the trajectory is an hyperbola, I do not comprehend its conjugate hyperbola under the name of this trajectory. For a body going on with a continued motion can never pass out of one hyperbola into its conjugate hyperbola.

The case when three points are given is more readily solved thus. Let B, C, D (Pl. 8, Fig. 3), be the given points. Join BC, CD, and produce them to E, F, so as EB may be to EC as SB to SC; and FC to FD as SC to SD. On EF drawn and produced let fall the perpendiculars SG, BH, and in GS produced indefinitely take GA to AS, and Ga to aS, as HB is to BS; then A will be the vertex, and Aa the principal axis of the trajectory: which, according as GA is greater than, equal to, or less than AS, will be either an ellipsis, a parabola, or an hyperbola; the point a in the first case falling on the same side of the line GF as the point A; in the second, going off to an infinite distance; in the third, falling on the other side of the line GF. For if on GF the perpendiculars CI, DK are let fall, IC will be to HB as EC to EB; that is, as SC to SB; and by permutation, IC to SC as HB to SB, or as GA to SA. And, by the like argument, we may prove that KD is to SD in the same ratio. Wherefore the points B, C, D lie in a conic section described about the focus S, in

fuch manner that all the right lines drawn from the focus S, to the feveral points of the fection, and the perpendiculars let fall from the fame points on the right line GF, are in that given ratio.

That excellent geometer M. *De la Hire* has folved this problem much after the fame way, in his Conics, prop. 25, lib. 8.

SECTION V.

How the orbits are to be found when neither focus is given.

LEMMA XVII.

If from any point P *of a given conic fection, to the four produced fides* AB, CD, AC, DB *of any trapezium* ABDC *infcribed in that fection, as many right lines* PQ, PR, PS, PT *are drawn in given angles, each line to each fide; the rectangle* PQ × PR *of thofe on the oppofite fides* AB, CD, *will be to the rectangle* PS × PT *of thofe on the other two oppofite fides* AC, BD, *in a given ratio.*

Case 1. Let us fuppofe, firft, that the lines drawn to one pair of oppofite fides are parallel to either of the other fides; as PQ and PR (Pl. 8, Fig. 4) to the fide AC, and PS and PT to the fide AB. And farther, that one pair of the oppofite fides, as AC and BD, are parallel betwixt themfelves; then the right line which bifects thofe parallel fides will be one of the diameters of the conic fection, and will likewife bifect RQ. Let O be the point in which RQ is bifected, and PO will be an ordinate to that diameter. Produce PO to K, fo that OK may be equal to PO, and OK will be an ordinate on the other fide of that diameter. Since, therefore, the points A, B, P, and K are placed in the conic fection, and PK cuts AB in a given angle, the rectangle PQK (by prop. 17, 19, 21, & 23, book 3, of *Apollonius*'s Conics) will be to the rectangle AQB in a given ratio. But QK and PR are equal, as being the differences of the equal lines OK, OP, and OQ, OR; whence the rectangles PQK and PQ × PR are equal; and therefore the rectangle PQ × PR is to the rectangle AQB, that is, to the rectangle PS × PT in a given ratio. Q.E.D.

Case 2. Let us next fuppofe that the oppofite fides AC and BD (Pl. 8, Fig. 5) of the trapezium are not parallel.

Draw Bd parallel to AC, and meeting as well the right line ST in t, as the conic section in d. Join Cd cutting PQ in r, and draw DM parallel to PQ, cutting Cd in M, and AB in N. Then (because of the similar triangles BTt, DBN), Bt or PQ is to Tt as DN to NB. And so Rr is to AQ or PS as DM to AN. Wherefore, by multiplying the antecedents by the antecedents, and the consequents by the consequents, as the rectangle PQ × Rr is to the rectangle PS × Tt, so will the rectangle NDM be to the rectangle ANB; and (by case 1) so is the rectangle PQ × Pr to the rectangle PS × Pt; and by division, so is the rectangle PQ × PR to the rectangle PS × PT. Q.E.D.

CASE 3. Let us suppose, lastly, the four lines PQ, PR, PS, PT (Pl. 8, Fig. 6), not to be parallel to the sides AC, AB, but any way inclined to them. In their place draw Pq, Pr parallel to AC; and Ps, Pt parallel to AB; and because the angles of the triangles PQq, PRr, PSs, PTt are given, the ratios of PQ to Pq, PR to Pr, PS to Ps, PT to Pt will be also given; and therefore the compounded ratios PQ × PR to Pq × Pr, and PS × PT to Ps × Pt are given. But from what we have demonstrated before, the ratio of Pq × Pr to Ps × Pt is given; and therefore also the ratio of PQ × PR to PS × PT. Q.E.D

LEMMA XVIII.

The same things supposed, if the rectangle PQ × PR *of the lines drawn to the two opposite sides of the trapezium is to the rectangle* PS × PT *of those drawn to the other two sides in a given ratio, the point* P, *from whence those lines are drawn, will be placed in a conic section described about the trapezium.* (Pl. 8, Fig. 7.)

Conceive a conic section to be described passing through the points A, B, C, D, and any one of the infinite number of points P, as for example p; I say, the point P will be always placed in this section. If you deny the thing, join AP cutting this conic section somewhere else, if possible, than in P, as in b. Therefore if from those points p and b, in the given angles to the sides of the trapezium, we draw the right lines pq, pr, ps, pt, and bk, bn, bf, bd, we shall have, as bk × bn to bf × bd, so (by lem. 17) pq × pr to ps × pt; and so (by supposition) PQ × PR to PS × PT. And because of the similar trapezia

bkAf, PQAS, as bk to bf, fo PQ to PS. Wherefore by dividing the terms of the preceding proportion by the correfpondent terms of this, we fhall have bn to bd as PR to PT. And therefore the equiangular trapezia Dn bd, DRPT are fimilar, and confequently their diagonals Db, DP do coincide. Wherefore b falls in the interfection of the right lines AP, DP, and confequently coincides with the point P. And therefore the point P, wherever it is taken, falls to be in the affigned conic fection. Q.E.D.

Cor. Hence if three right lines PQ, PR, PS, are drawn from a common point P, to as many other right lines given in pofition, AB, CD, AC, each to each, in as many angles refpectively given, and the rectangle PQ × PR under any two of the lines drawn be to the fquare PS^2 of the third in a given ratio; the point P, from which the right lines are drawn, will be placed in a conic fection that touches the lines AB, CD in A and C; and the contrary. For the pofition of the three right lines AB, CD, AC remaining the fame, let the line BD approach to and coincide with the line AC; then let the line PT come likewife to coincide with the line PS; and the rectangle PS × PT will become PS^2, and the right lines AB, CD, which before did cut the curve in the points A and B, C and D, can no longer cut, but only touch, the curve in thofe coinciding points.

SCHOLIUM.

In this lemma, the name of conic fection is to be underftood in a large fenfe, comprehending as well the rectilinear fection through the vertex of the cone, as the circular one parallel to the bafe. For if the point p happens to be in a right line, by which the points A and D, or C and B are joined, the conic fection will be changed into tworight lines, one of which is that right line upon which the point p falls, and the other is a right line that joins the other two of the four points. If the two oppofite angles of the trapezium taken together are equal to two right angles, and if the four lines PQ, PR, PS, PT are drawn to the fides thereof at right angles, or any other equal angles, and the rectangle PQ × PR under two of the lines drawn PQ and PR, is equal to the rectangle PS × PT under

the other two PS and PT, the conic ſection will become a circle. And the ſame thing will happen if the four lines are drawn in any angles, and the rectangle PQ × PR, under one pair of the lines drawn, is to the rectangle PS × PT under the other pair as the rectangle under the ſines of the angles S, T, in which the two laſt lines PS, PT are drawn to the rectangle under the ſines of the angles Q, R, in which the two firſt PQ, PR are drawn. In all other caſes the locus of the point P will be one of the three figures which paſs commonly by the name of the conic ſections. But in room of the trapezium ABCD, we may ſubſtitute a quadrilateral figure whoſe two oppoſite ſides croſs one another like diagonals. And one or two of the four points A, B, C, D may be ſuppoſed to be removed to an infinite diſtance, by which means the ſides of the figure which converge to thoſe points, will become parallel: and in this caſe the conic ſection will paſs through the other points, and will go the ſame way as the parallels *in infinitum.*

LEMMA XIX.

To find a point P (Pl. 8, Fig. 8) *from which if four right lines* PQ, PR, PS, PT *are drawn to as many other right lines* AB, CD, AC, BD *given by poſition, each to each, at given angles, the rectangle* PQ × PR, *under any two of the lines drawn, ſhall be to the rectangle* PS × PT, *under the other two, in a given ratio.*

Suppoſe the lines AB, CD, to which the two right lines PQ, PR, containing one of the rectangles, are drawn to meet two other lines, given by poſition, in the points A, B, C, D. From one of thoſe, as A, draw any right line AH, in which you would find the point P. Let this cut the oppoſite lines BD, CD, in H and I; and, becauſe all the angles of the figure are given, the ratio of PQ to PA, and PA to PS, and therefore of PQ to PS, will be alſo given. Subducting this ratio from the given ratio of PQ × PR to PS × PT, the ratio of PR to PT will be given; and adding the given ratios of PI to PR, and PT to PH, the ratio of PI to PH, and therefore the point P will be given. Q.E.I.

Cor. 1. Hence alſo a tangent may be drawn to any point D of the locus of all the points P. For the chord PD, where

the points P and D meet, that is, where AH is drawn through the point D, becomes a tangent. In which case the ultimate ratio of the evanescent lines IP and PH will be found as above. Therefore draw CF parallel to AD, meeting BD in F, and cut it in E in the same ultimate ratio, then DE will be the tangent; because CF and the evanescent IH are parallel, and similarly cut in E and P.

COR. 2. Hence also the locus of all the points P may be determined. Through any of the points A, B, C, D, as A, (Pl. 9, Fig. 1), draw AE touching the locus, and through any other point B parallel to the tangent, draw BF meeting the locus in F; and find the point F by this lemma. Bisect BF in G, and, drawing the indefinite line AG, this will be the position of the diameter to which BG and FG are ordinates. Let this AG meet the locus in H, and AH will be its diameter or latus transversum, to which the latus rectum will be as BG^2 to $AG \times GH$. If AG no where meets the locus, the line AH being infinite, the locus will be a parabola; and its latus rectum corresponding to the diameter AG will be $\frac{BG^2}{AG}$. But if it does meet it any where, the locus will be an hyperbola, when the points A and H are placed on the same side the point G; and an ellipsis, if the point G falls between the points A and H; unless, perhaps, the angle AGB is a right angle, and at the same time BG^2 equal to the rectangle AGH, in which case the locus will be a circle.

And so we have given in this corollary a solution of that famous problem of the antients concerning four lines, begun by *Euclid*, and carried on by *Apollonius*; and this not an analytical calculus, but a geometrical composition, such as the antients required.

LEMMA XX.

If the two opposite angular points A *and* P (Pl. 9, Fig. 2) *of any parallelogram* ASPQ *touch any conic section in the points* A *and* P; *and the sides* AQ, AS *of one of those angles, indefinitely produced, meet the same conic section in* B *and* C; *and from the points of concourse* B *and* C *to any fifth point* D *of the conic section, two right lines* BD, CD

are drawn meeting the two other sides PS, PQ *of the parallelogram, indefinitely produced in* T *and* R; *the parts* PR *and* PT, *cut off from the sides, will always be one to the other in a given ratio. And* vice versa, *if those parts cut off are one to the other in a given ratio, the locus of the point* D *will be a conic section passing through the four points* A, B, C, P.

CASE 1. Join BP, CP, and from the point D draw the two right lines DG, DE, of which the first DG shall be parallel to AB, and meet PB, PQ, CA in H, I, G; and the other DE shall be parallel to AC, and meet PC, PS, AB, in F, K, E; and (by lem. 17) the rectangle DE × DF will be to the rectangle DG × DH in a given ratio. But PQ is to DE (or IQ) as PB to HB, and consequently as PT to DH; and by permutation PQ is to PT as DE to DH. Likewise PR is to DF as RC to DC, and therefore as (IG or) PS to DG; and by permutation PR is to PS as DF to DG; and, by compounding those ratios, the rectangle PQ × PR will be to the rectangle PS × PT as the rectangle DE × DF is to the rectangle DG × DH, and consequently in a given ratio. But PQ and PS are given, and therefore the ratio of PR to PT is given. Q.E.D.

CASE 2. But if PR and PT are supposed to be in a given ratio one to the other, then by going back again, by a like reasoning, it will follow that the rectangle DE × DF is to the rectangle DG × DH in a given ratio; and so the point D (by lem. 18) will lie in a conic section passing through the points A, B, C, P, as its locus. Q.E.D.

COR. 1. Hence if we draw BC cutting PQ in r, and in PT take Pt to Pr in the same ratio which PT has to PR; then Bt will touch the conic section in the point B. For suppose the point D to coalesce with the point B, so that the chord BD vanishing, BT shall become a tangent, and CD and BT will coincide with CB and Bt.

COR. 2. And, vice versa, if Bt is a tangent, and the lines BD, CD meet in any point D of a conic section, PR will be to PT as Pr to Pt. And, on the contrary, if PR is to PT as Pr to Pt, then BD and CD will meet in some point D of a conic section.

COR. 3. One conic section cannot cut another conic section in more than four points. For, if it is possible, let two conic sections pass through the five points A, B, C, P, O; and let the right line BD cut them in the points D, d, and the right line Cd cut the right line PQ in q. Therefore PR is to PT as Pq to PT: whence PR and Pq are equal one to the other, against the supposition.

LEMMA XXI.

If two moveable and indefinite right lines BM, CM *drawn through given points* B, C, *as poles, do by their point of concourse* M *describe a third right line* MN *given by position; and other two indefinite right lines* BD, CD *are drawn, making with the former two at those given points* B, C, *given angles,* MBD, MCD: *I say, that those two right lines* BD, CD *will by their point of concourse* D *describe a conic section passing through the points* B, C. *And,* vice versa, *if the right lines* BD, CD *do by their point of concourse* D *describe a conic section passing through the given points* B, C, A, *and the angle* DBM *is always equal to the given angle* ABC, *as well as the angle* DCM *always equal to the given angle* ACB, *the point* M *will lie in a right line given by position, as its locus.* (Pl. 9, Fig. 3.)

For in the right line MN let a point N be given, and when the moveable point M falls on the immovable point N, let the moveable point D fall on an immovable point P. Join CN, BN, CP, BP, and from the point P draw the right lines PT, PR meeting BD, CD in T and R, and making the angle BPT equal to the given angle BNM, and the angle CPR equal to the given angle CNM. Wherefore since (by supposition) the angles MBD, NBP are equal, as also the angles MCD, NCP, take away the angles NBD and NCD that are common, and there will remain the angles NBM and PBT, NCM and PCR equal; and therefore the triangles NBM, PBT are similar, as also the triangles NCM, PCR. Wherefore PT is to NM as PB to NB; and PR to NM as PC to NC. But the points B, C, N, P are immovable: wherefore PT and PR have a given ratio to NM, and consequently a given ratio between themselves; and therefore, (by lemma 20) the point D wherein the moveable right

lines BT and CR perpetually concur, will be placed in a conic section passing through the points B, C, P. Q E.D.

And, *vice versa*, if the moveable point D (Pl. 9, Fig. 4) lies in a conic section passing through the given points B, C, A; and the angle DBM is always equal to the given angle ABC, and the angle DCM always equal to the given angle ACB, and when the point D falls successively on any two immovable points p, P, of the conic section, the moveable point M falls successively on two immovable points n, N. Through these points n, N, draw the right line nN: this line nN will be the perpetual locus of that moveable point M. For, if possible, let the point M be placed in any curve line. Therefore the point D will be placed in a conic section passing through the five points B, C, A, p, P, when the point M is perpetually placed in a curve line. But from what was demonstrated before, the point D will be also placed in a conic section passing through the same five points B, C, A, p, P, when the point M is perpetually placed in a right line. Wherefore the two conic sections will both pass through the same five points, against corol. 3, lem. 20. It is therefore absurd to suppose that the point M is placed in a curve line. Q.E.D.

PROPOSITION XXII. PROBLEM XIV.

To describe a trajectory that shall pass through five given points. (Pl. 9, Fig. 5.)

Let the five given points be A, B, C, P, D. From any one of them, as A, to any other two as B, C, which may be called the poles, draw the right lines AB, AC, and parallel to those the lines TPS, PRQ through the fourth point P. Then from the two poles B, C, draw through the fifth point D two indefinite lines BDT, CRD, meeting with the last drawn lines TPS, PRQ (the former with the former, and the latter with the latter) in T and R. Then drawing the right line tr parallel to TR, cutting off from the right lines PT, PR, any segments Pt, Pr, proportional to PT, PR; and if through their extremities t, r, and the poles B, C, the right lines Bt, Cr are drawn, meeting in d, that point d will be placed in the trajectory required. For (by lem. 20) that point d is placed in a conic section passing through the four points A, B,

C, P; and the lines Rr, Tt vanishing, the point d comes to coincide with the point D. Wherefore the conic section passes through the five points A, B, C, P, D. Q.E.D.

The same otherwise. (Pl. 9, Fig. 6.)

Of the given points join any three, as A, B, C; and about two of them B, C, as poles, making the angles ABC, ACB of a given magnitude to revolve, apply the legs BA, CA, first to the point D, then to the point P, and mark the points M, N, in which the other legs BL, CL interfect each the other in both cases. Draw the indefinite right line MN, and let those moveable angles revolve about their poles B, C, in such manner that the intersection, which is now supposed to be m, of the legs BL, CL, or BM, CM, may always fall in that indefinite right line MN; and the intersection, which is now supposed to be d, of the legs BA, CA, or BD, CD, will describe the trajectory required, PADdB. For (by lem. 21) the point d will be placed in a conic section passing through the points B, C; and when the point m comes to coincide with the points L, M, N, the point d will (by construction) come to coincide with the points A, D, P. Wherefore a conic section will be described that shall pass through the five points A, B, C, P, D. Q.E.F.

Cor. 1. Hence a right line may be readily drawn which shall be a tangent to the trajectory in any given point B. Let the point d come to coincide with the point B, and the right line Bd will become the tangent required.

Cor. 2. Hence also may be found the centres, diameters, and latera recta of the trajectories, as in cor. 2, lem. 19.

SCHOLIUM.

The former of these constructions (Fig. 5) will become something more simple by joining BP, and in that line, produced, if need be, taking Bp to BP as PR is to PT; and through p draw the indefinite right line pe parallel to S P; and in that line pe taking always pe equal to Pr; and draw the right lines Be, Cr to meet in d. For since Pr to Pt, PR to PT, pB to PB, pe to Pt, are all in the same ratio, pe and Pr will be always equal. After this manner the points of

the trajectory are most readily found, unless you would rather describe the curve mechanically, as in the second construction.

PROPOSITION XXIII. PROBLEM XV.

To describe a trajectory that shall pass through four given points, and touch a right line given by position. (Pl. 10, Fig. 1.)

CASE 1. Suppose that HB is the given tangent, B the point of contact, and C, D, P, the three other given points. Join BC, and draw PS parallel to BH, and PQ parallel to BC; complete the parallelogram BSPQ. Draw BD cutting SP in T, and CD cutting PQ in R. Lastly, draw any line tr parallel to TR, cutting off from PQ, PS, the segments Pr, Pt proportional to PR, PT respectively; and draw Cr, Bt, their point of concourse d will (by lem. 20) always fall on the trajectory to be described.

The same otherwise. (Pl. 10, Fig. 2.)

Let the angle CBH of a given magnitude revolve about the pole B, as also the rectilinear radius DC, both ways produced, about the pole C. Mark the points M, N, on which the leg BC of the angle cuts that radius when BH, the other leg thereof, meets the same radius in the points P and D. Then drawing the indefinite line MN, let that radius CP or CD and the leg BC of the angle perpetually meet in this line; and the point of concourse of the other leg BH with the radius will delineate the trajectory required.

For if in the constructions of the preceding problem the point A comes to a coincidence with the point B, the lines CA and CB will coincide, and the line AB, in its last situation, will become the tangent BH; and therefore the constructions there set down will become the same with the constructions here described. Wherefore the concourse of the leg BH with the radius will describe a conic section passing through the points C, D, P, and touching the line BH in the point B. Q.E.F.

CASE 2. Suppose the four points B, C, D, P (Pl. 10, Fig. 3), given, being situated without the tangent HI. Join each two by the lines BD, CP, meeting in G, and cutting the tangent in H and I. Cut the tangent in A in such man-

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ner that HA may be to IA as the rectangle under a mean proportional between CG and GP, and a mean proportional between BH and HD is to a rectangle under a mean proportional between GD and GB, and a mean proportional between PI and IC, and A will be the point of contact. For if HX, a parallel to the right line PI, cuts the trajectory in any points X and Y; the point A (by the properties of the conic sections) will come to be so placed, that HA^2 will become to AI^2 in a ratio that is compounded out of the ratio of the rectangle XHY to the rectangle BHD, or of the rectangle CGP to the rectangle DGB; and the ratio of the rectangle BHD to the rectangle PIC. But after the point of contact A is found, the trajectory will be described as in the first case. Q.E.F. But the point A may be taken either between or without the points H and I; upon which account a twofold trajectory may be described.

PROPOSITION XXIV. PROBLEM XVI.

To describe a trajectory that shall pass through three given points, and touch two right lines given by position. (Pl. 10, Fig. 4.)

Suppose HI, KL to be the given tangents, and B, C, D, the given points. Through any two of those points, as B, D, draw the indefinite right line BD meeting the tangents in the points H, K. Then likewise through any other two of these points, as C, D, draw the indefinite right line CD meeting the tangents in the points I, L. Cut the lines drawn in R and S, so that HR may be to KR as the mean proportional between BH and HD is to the mean proportional between BK and KD; and IS to LS as the mean proportional between CI and ID is to the mean proportional between CL and LD. But you may cut, at pleasure, either within or between the points K and H, I and L, or without them; then draw RS cutting the tangents in A and P, and A and P will be the points of contact. For if A and P are supposed to be the points of contact, situated any where else in the tangents, and through any of the points H, I, K, L, as I, situated in either tangent HI, a right line IY is drawn parallel to the other tangent KL, and meeting the curve in X and Y, and in that right line there be taken IZ equal to a mean proportional between IX and IY, the rectangle XIY or IZ^2,

will (by the properties of the conic sections) be to LP^2 as the rectangle CID is to the rectangle CLD, that is (by the construction), as SI^2 is to SL^2, and therefore IZ is to LP as SI to SL. Wherefore the points S, P, Z, are in one right line. Moreover, since the tangents meet in G, the rectangle XIY or IZ^2 will (by the properties of the conic sections) be to IA^2 as GP^2 is to GA^2, and consequently IZ will be to IA as GP to GA. Wherefore the points P, Z, A, lie in one right line, and therefore the points S, P, and A are in one right line. And the same argument will prove that the points R, P, and A are in one right line. Wherefore the points of contact A and P lie in the right line RS. But after these points are found, the trajectory may be described, as in the first case of the preceding problem. Q.E.F.

In this proposition, and case 2 of the foregoing, the constructions are the same, whether the right line XY cut the trajectory in X and Y, or not; neither do they depend upon that section. But the constructions being demonstrated where that right line does cut the trajectory, the constructions where it does not are also known; and therefore, for brevity's sake, I omit any farther demonstration of them.

LEMMA XXII.

To transform figures into other figures of the same kind. (Pl. 10. Fig. 5.)

Suppose that any figure HGI is to be transformed. Draw, at pleasure, two parallel lines AO, BL, cutting any third line AB, given by position, in A and B, and from any point G of the figure, draw out any right line GD, parallel to OA, till it meet the right line AB. Then from any given point O in the line OA, draw to the point D the right line OD, meeting BL in d; and from the point of concourse raise the right line dg containing any given angle with the right line BL, and having such ratio to Od as DG has to OD; and g will be the point in the new figure hgi, corresponding to the point G. And in like manner the several points of the first figure will give as many correspondent points of the new figure. If we therefore conceive the point G to be carried along by a continual motion through all the points of the first figure, the point g will be likewise carried along by a continual motion

through all the points of the new figure, and describe the same. For distinction's sake, let us call DG the first ordinate, dg the new ordinate, AD the first abscissa, ad the new abscissa; O the pole, OD the abscinding radius, OA the first ordinate radius, and Oa (by which the parallelogram OABa is completed) the new ordinate radius.

I say, then, that if the point G is placed in a right line given by position, the point g will be also placed in a right line given by position. If the point G is placed in a conic section, the point g will be likewise placed in a conic section. And here I understand the circle as one of the conic sections. But farther, if the point G is placed in a line of the third analytical order, the point g will also be placed in a line of the third order, and so on in curve lines of higher orders. The two lines in which the points G, g, are placed, will be always of the same analytical order. For as ad is to OA, so are od to OD, dg to DG, and AB to AD; and therefore AD is equal to $\frac{OA \times AB}{ad}$, and DG equal to $\frac{OA \times dg}{ad}$. Now if the point G is placed in a right line, and therefore, in any equation by which the relation between the abscissa AD and the ordinate DG is expressed, those indetermined lines AD and DG rise no higher than to one dimension, by writing this equation $\frac{OA \times AB}{ad}$ in place of AD, and $\frac{OA \times dg}{ad}$ in place of DG, a new equation will be produced, in which the new abscissa ad and new ordinate dg rise only to one dimension; and which therefore must denote a right line. But if AD and DG (or either of them) had risen to two dimensions in the first equation, ad and dg would likewise have risen to two dimensions in the second equation. And so on in three or more dimensions. The indetermined lines, ad, dg in the second equation, and AD, DG, in the first, will always rise to the same number of dimensions; and therefore the lines in which the points G, g, are placed are of the same analytical order.

I say farther, that if any right line touches the curve line in the first figure, the same right line transferred the same way with the curve into the new figure will touch that curve line in the new figure, and *vice versa*. For if any two points of the

curve in the firſt figure are ſuppoſed to approach one the other till they come to coincide, the ſame points transferred will approach one the other till they come to coincide in the new figure; and therefore the right lines with which thoſe points are joined will become together tangents of the curves in both figures. I might have given demonſtrations of theſe aſſertions in a more geometrical form; but I ſtudy to be brief.

Wherefore if one rectilinear figure is to be transformed into another, we need only transfer the interſections of the right lines of which the firſt figure conſiſts, and through the tranſferred interſections to draw right lines in the new figure. But if a curvilinear figure is to be transformed, we muſt transfer the points, the tangents, and other right lines, by means of which the curve line is defined. This lemma is of uſe in the ſolution of the more difficult problems; for thereby we may tranſform the propoſed figures, if they are intricate, into others that are more ſimple. Thus any right lines converging to a point are transformed into parallels, by taking for the firſt ordinate radius any right line that paſſes through the point of concourſe of the converging lines, and that becauſe their point of concourſe is by this means made to go off *in infinitum;* and parallel lines are ſuch as tend to a point infinitely remote. And after the problem is ſolved in the new figure, if by the inverſe operations we transform the new into the firſt figure, we ſhall have the ſolution required.

This lemma is alſo of uſe in the ſolution of ſolid problems. For as often as two conic ſections occur, by the interſection of which a problem may be ſolved, any one of them may be transformed, if it is an hyperbola or a parabola, into an ellipſis, and then this ellipſis may be eaſily changed into a circle. So alſo a right line and a conic ſection, in the conſtruction of plane problems, may be transformed into a right line and a circle.

PROPOSITION XXV. PROBLEM XVII.

To deſcribe a trajectory that ſhall paſs through two given points, and touch three right lines given by poſition. (Pl. 10, Fig. 6.)

Through the concourse of any two of the tangents one with the other, and the concourse of the third tangent with the right line which passes through the two given points, draw an indefinite right line; and, taking this line for the first ordinate radius, transform the figure by the preceding lemma into a new figure. In this figure those two tangents will become parallel to each other, and the third tangent will be parallel to the right line that passes through the two given points. Suppose hi, kl to be those two parallel tangents, ik the third tangent, and hl a right line parallel thereto, passing through those points a, b, through which the conic section ought to pass in this new figure; and completing the parallelogram hikl, let the right lines hi, ik, kl be so cut in c, d, e, that hc may be to the square root of the rectangle ahb, ic to id, and ke to kd, as the sum of the right lines hi and kl is to the sum of the three lines, the first whereof is the right line ik, and the other two are the square roots of the rectangles ahb and alb; and c, d, e, will be the points of contact. For by the properties of the conic sections, hc^2 to the rectangle ahb, and ic^2 to id^2, and ke^2 to kd^2, and el^2 to the rectangle alb, are all in the same ratio; and therefore hc to the square root of ahb, ic to id, ke to kd, and el to the square root of alb, are in the subduplicate of that ratio; and by composition, in the given ratio of the sum of all the antecedents $hi+kl$, to the sum of all the consequents $\sqrt{ahb}+ik+\sqrt{alb}$. Wherefore from that given ratio we have the points of contact c, d, e, in the new figure. By the inverted operations of the last lemma, let those points be transferred into the first figure, and the trajectory will be there described by prob. 14. Q.E.F. But according as the points a, b, fall between the points h, l, or without them, the points c, d, e, must be taken either between the points, h, i, k, l, or without them. If one of the points a, b, falls between the points h, l, and the other without the points h, l, the problem is impossible.

PROPOSITION XXVI. PROBLEM. XVIII.

To describe a trajectory that shall pass through a given point, and touch four right lines given by position. (Pl. 11, Fig. 1.)

From the common interfections of any two of the tangents to the common interfection of the other two, draw an indefinite right line; and taking this line for the firft ordinate radius, transform the figure (by lem. 22) into a new figure, and the two pairs of tangents, each of which before concurred in the firft ordinate radius, will now become parallel. Let hi and kl, ik and hl, be thofe pairs of parallels completing the parallelogram hikl. And let p be the point in this new figure correfponding to the given point in the firft figure. Through O the centre of the figure draw pq; and Oq being equal to Op, q will be the other point through which the conic fection muft pafs in this new figure. Let this point be transferred, by the inverfe operation of lem. 22 into the firft figure, and there we fhall have the two points through which the trajectory is to be defcribed. But through thofe points that trajectory may be defcribed by prob. 17. Q.E.F.

LEMMA XXIII.

If two right lines, as AC, BD *given by pofition, and terminating in given points* A, B, *are in a given ratio one to the other, and the right line* CD, *by which the indetermined points* C, D *are joined, is cut in* K *in a given ratio; I fay, that the point* K *will be placed in a right line given by pofition.* (Pl. 11, Fig. 2.)

For let the right lines AC, BD meet in E, and in BE take BG to AE as BD is to AC, and let FD be always equal to the given line EG; and, by conftruction, EC will be to GD, that is, to EF, as AC to BD, and therefore in a given ratio; and therefore the triangle EFC will be given in kind. Let CF be cut in L fo as CL may be to CF in the ratio of CK to CD; and becaufe that is a given ratio, the triangle EFL will be given in kind, and therefore the point L will be placed in the right line EL given by pofition. Join LK, and the triangles CLK, CFD will be fimilar; and becaufe FD is a given line, and LK is to FD in a given ratio, LK will be alfo given. To this let EH be taken equal, and ELKH will be always a parallelogram. And therefore the point K is always placed in the fide HK (given by pofition) of that parallelogram. Q.E.D.

g. 1.

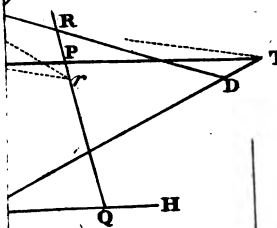
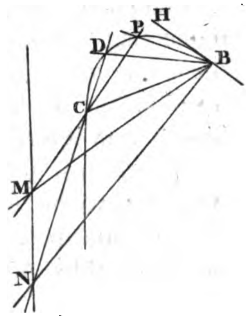


Fig. 2.



g. 3.

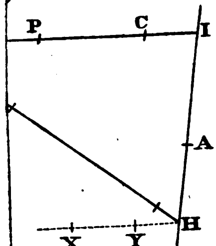
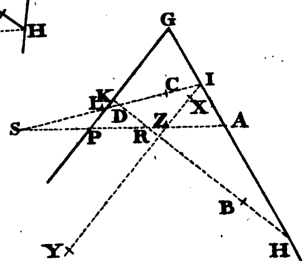


Fig. 4.



5.

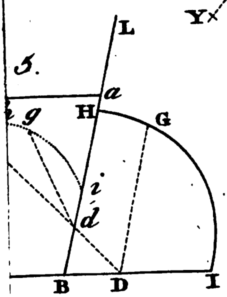
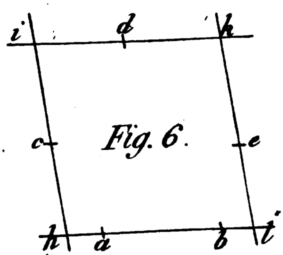


Fig. 6.



Cor. Becaufe the figure EFLC is given in kind, the three right lines EF, EL, and EC, that is, GD, HK, and EC, will have given ratios to each other.

LEMMA XXIV.

If three right lines, two whereof are parallel, and given by pofition, touch any conic fection; I fay, that the femi-diameter of the fection which is parallel to thofe two is a mean proportional between the fegments of thofe two that are intercepted between the points of contact and the third tangent. (Pl. 11, Fig. 3.)

Let AF, GB be the two parallels touching the conic fection ADB in A and B; EF the third right line touching the conic fection in I, and meeting the two former tangents in F and G, and let CD be the femi-diameter of the figure parallel to thofe tangents; I fay, that AF, CD, BG are continually proportional.

For if the conjugate diameters AB, DM meet the tangent FG in E and H, and cut one the other in C, and the parallelogram IKCL be completed; from the nature of the conic fections, EC will be to CA as CA to CL; and fo by divifion, EC — CA to CA — CL, or EA to AL; and by compofition, EA to EA + AL or EL, as EC to EC + CA or EB; and therefore (becaufe of the fimilitude of the triangles EAF, ELI, ECH, EBG) AF is to LI as CH to BG. Likewife, from the nature of the conic fections, LI (or CK) is to CD as CD to CH; and therefore (*ex æquo perturbate*) AF is to CD as CD to BG. Q.E.D.

Cor. 1. Hence if two tangents FG, PQ meet two parallel tangents AF, BG in F and G, P and Q, and cut one the other in O; AF (*ex æquo perturbate*) will be to BQ as AP to BG, and by divifion, as FP to GQ, and therefore as FO to OG.

Cor. 2. Whence alfo the two right lines PG, FQ drawn through the points P and G, F and Q, will meet in the right line ACB paffing through the centre of the figure and the points of contact A, B.

LEMMA XXV.

If four sides of a parallelogram indefinitely produced touch any conic section, and are cut by a fifth tangent; I say, that, taking those segments of any two conterminous sides that terminate in opposite angles of the parallelogram, either segment is to the side from which it is cut off as that part of the other conterminous side which is intercepted between the point of contact and the third side is to the other segment. (Pl. 11, Fig. 4.)

Let the four sides ML, IK, KL, MI of the parallelogram MLIK touch the conic section in A, B, C, D; and let the fifth tangent FQ cut those sides in F, Q, H, and E; and taking the segments ME, KQ of the sides MI, KI, or the segments KH, MF of the sides KL, ML, I say, that ME is to MI as BK to KQ; and KH to KL as AM to MF. For, by cor. 1 of the preceding lemma, ME is to EI as (AM or) BK to BQ; and, by composition, ME is to MI as BK to KQ. Q.E.D. Also KH is to HL as (BK or) AM to AF; and by division, KH to KL as AM to MF. Q.E.D.

Cor. 1. Hence if a parallelogram IKLM described about a given conic section is given, the rectangle KQ × ME, as also the rectangle KH × MF equal thereto, will be given. For, by reason of the similar triangles KQH, MFE, those rectangles are equal.

Cor. 2. And if a sixth tangent eq is drawn meeting the tangents KI, MI in q and e, the rectangle KQ × ME will be equal to the rectangle Kq × Me, and KQ will be to Me as Kq to ME, and by division as Qq to Ee.

Cor. 3. Hence, also, if Eq, eQ, are joined and bisected, and a right line is drawn through the points of bisection, this right line will pass through the centre of the conic section. For since Qq is to Ee as KQ to Me, the same right line will pass through the middle of all the lines Eq, eQ, MK (by lem. 22), and the middle point of the right line MK is the centre of the section.

PROPOSITION XXVII. PROBLEM XIX.

To describe a trajectory that may touch five right lines given by position. (Pl. 11, Fig. 5.)

Suppoſing ABG, BCF, GCD, FDE, EA to be the tangents given by poſition. Biſect in M and N, AF, BE, the diagonals of the quadrilateral figure ABFE contained under any four of them; and (by cor. 3, lem. 25) the right line MN drawn through the points of biſection will paſs through the centre of the trajectory. Again, biſect in P and Q the diagonals (if I may ſo call them) BD, GF of the quadrilateral figure BGDF contained under any other four tangents, and the right line PQ drawn through the points of biſection will paſs through the centre of the trajectory; and therefore the centre will be given in the concourſe of the biſecting lines. Suppoſe it to be O. Parallel to any tangent BC draw KL at ſuch diſtance that the centre O may be placed in the middle between the parallels; this KL will touch the trajectory to be deſcribed. Let this cut any other two tangents GCD, FDE, in L and K. Through the points C and K, F and L, where the tangents not parallel, CL, FK meet the parallel tangents CF, KL, draw CK, FL meeting in R; and the right line OR drawn and produced, will cut the parallel tangents CF, KL, in the points of contact. This appears from cor. 3, lem. 24. And by the ſame method the other points of contact may be found, and then the trajectory may be deſcribed by prob. 14. Q.E.F.

SCHOLIUM.

Under the preceding propoſitions are comprehended thoſe problems wherein either the centres or aſymptotes of the trajectories are given. For when points and tangents and the centre are given, as many other points and as many other tangents are given at an equal diſtance on the other ſide of the centre. And an aſymptote is to be conſidered as a tangent, and its infinitely remote extremity (if we may ſay ſo) is a point of contact. Conceive the point of contact of any tangent removed *in infinitum*, and the tangent will degenerate into an aſymptote, and the conſtructions of the preceding problems will be changed into the conſtructions of thoſe problems wherein the aſymptote is given.

After the trajectory is deſcribed, we may find its axes and foci in this manner. In the conſtruction and figure of lem.

21, (Pl. 12, Fig. 1), let thofe legs BP, CP, of the moveable angles PBN, PCN, by the concourfe of which the trajectory was defcribed, be made parallel one to the other; and retaining that pofition, let them revolve about their poles B, C, in that figure. In the mean while let the other legs CN, BN, of thofe angles, by their concourfe K or k, defcribe the circle BKGC. Let O be the centre of this circle; and from this centre upon the ruler MN, wherein thofe legs CN, BN did concur while the trajectory was defcribed, let fall the perpendicular OH meeting the circle in K and L. And when thofe other legs CK, BK meet in the point K that is neareft to the ruler, the firft legs CP, BP will be parallel to the greater axis, and perpendicular on the leffer; and the contrary will happen if thofe legs meet in the remoteft point L. Whence if the centre of the trajectory is given, the axes will be given; and thofe being given, the foci will be readily found.

But the fquares of the axes are one to the other as KH to LH, and thence it is eafy to defcribe a trajectory given in kind through four given points. For if two of the given points are made the poles C, B, the third will give the moveable angles PCK, PBK; but thofe being given, the circle BGKC may be defcribed. Then, becaufe the trajectory is given in kind, the ratio of OH to OK, and therefore OH itfelf, will be given. About the centre O, with the interval OH, defcribe another circle, and the right line that touches this circle, and paffes through the concourfe of the legs CK, BK, when the firft legs CP, BP meet in the fourth given point, will be the ruler MN, by means of which the trajectory may be defcribed. Whence alfo on the other hand a trapezium given in kind (excepting a few cafes that are impoffible) may be infcribed in a given conic fection.

There are alfo other lemmas, by the help of which trajectories given in kind may be defcribed through given points, and touching given lines. Of fuch a fort is this, that if a right line is drawn through any point given by pofition, that may cut a given conic fection in two points, and the diftance of the interfections is bifected, the point of bifection will touch another conic fection of the fame kind with the former, and

having its axes parallel to the axes of the former. But I hasten to things of greater use.

LEMMA XXVI.

To place the three angles of a triangle, given both in kind and magnitude, in respect of as many right lines given by position, provided they are not all parallel among themselves, in such manner that the several angles may touch the several lines. (Pl. 12, Fig. 2.)

Three indefinite right lines AB, AC, BC, are given by position, and it is required so to place the triangle DEF that its angle D may touch the line AB, its angle E the line AC, and its angle F the line BC. Upon DE, DF, and EF, describe three segments of circles DRE, DGF, EMF, capable of angles equal to the angles BAC, ABC, ACB respectively. But those segments are to be described towards such sides of the lines DE, DF, EF, that the letters DRED may turn round about in the same order with the letters BACB; the letters DGFD in the same order with the letters ABCA; and the letters EMFE in the same order with the letters ACBA; then, completing those segments into entire circles, let the two former circles cut one the other in G, and suppose P and Q to be their centres. Then joining GP, PQ, take Ga to AB as GP is to PQ; and about the centre G, with the interval Ga, describe a circle that may cut the first circle DGE in a. Join aD cutting the second circle DFG in b, as well as aE cutting the third circle EMF in c. Complete the figure ABCdef similar and equal to the figure abcDEF: I say, the thing is done.

For drawing Fc meeting aD in n, and joining aG, bG, QG, QD, PD, by construction the angle EaD is equal to the angle CAB, and the angle acF equal to the angle ACB; and therefore the triangle anc equiangular to the triangle ABC. Wherefore the angle anc or FnD is equal to the angle ABC, and consequently to the angle FbD; and therefore the point n falls on the point b. Moreover the angle GPQ, which is half the angle GPD at the centre, is equal to the angle GaD at the circumference; and the angle GQP, which is half the angle GQD at the centre, is equal to the comple-

ment to two right angles of the angle GbD at the circumference, and therefore equal to the angle Gab. Upon which account the triangles GPQ, Gab, are fimilar, and Ga is to ab as GP to PQ; that is (by conftruction), as Ga to AB. Wherefore ab and AB are equal; and confequently the triangles abc, ABC, which we have now proved to be fimilar, are alfo equal. And therefore fince the angles D, E, F, of the triangle DEF do refpectively touch the fides ab, ac, bc of the triangle abc, the figure ABCdef may be completed fimilar and equal to the figure abcDEF, and by completing it the problem will be folved. Q.E.F.

Cor. Hence a right line may be drawn whofe parts given in length may be intercepted between three right lines given by pofition. Suppofe the triangle DEF, by the accefs of its point D to the fide EF, and by having the fides DE, DF placed *in directum* to be changed into a right line whofe given part DE is to be interpofed between the right lines AB, AC given by pofition; and its given part DF is to be interpofed between the right lines AB, BC, given by pofition; then, by applying the preceding conftruction to this cafe, the problem will be folved.

PROPOSITION XXVIII. PROBLEM XX.

To defcribe a trajectory given both in kind and magnitude, given parts of which fhall be interpofed between three right lines given by pofition. (Pl. 12, Fig 3.)

Suppofe a trajectory is to be defcribed that may be fimilar and equal to the curve line DEF, and may be cut by three right lines AB, AC, BC, given by pofition, into parts DE and EF, fimilar and equal to the given parts of this curve line.

Draw the right lines DE, EF, DF; and place the angles D, E, F, of this triangle DEF, fo as to touch thofe right lines given by pofition (by lem. 26). Then about the triangle defcribe the trajectory, fimilar and equal to the curve DEF. Q.E.F.

LEMMA XXVII.

To defcribe a trapezium given in kind, the angles whereof may be fo placed, in refpect of four right lines given by pofition,

that are neither all parallel among themſelves, nor converge to one common point, that the ſeveral angles may touch the ſeveral lines. (Pl. 13, Fig. 1.)

Let the four right lines ABC, AD, BD, CE, be given by poſition; the firſt cutting the ſecond in A, the third in B, and the fourth in C; and ſuppoſe a trapezium fghi is to be deſcribed that may be ſimilar to the trapezium FGHI, and whoſe angle f, equal to the given angle F, may touch the right line ABC; and the other angles g, h, i, equal to the other given angles, G, H, I, may touch the other lines AD, BD, CE, reſpectively. Join FH, and upon FG, FH, FI deſcribe as many ſegments of circles FSG, FTH, FVI, the firſt of which FSG may be capable of an angle equal to the angle BAD; the ſecond FTH capable of an angle equal to the angle CBD; and the third FVI of an angle equal to the angle ACE. But the ſegments are to be deſcribed towards thoſe ſides of the lines FG, FH, FI, that the circular order of the letters FSGF may be the ſame as of the letters BADB, and that the letters FTHF may turn about in the ſame order as the letters CBDC, and the letters FVIF in the ſame order as the letters ACEA. Complete the ſegments into entire circles, and let P be the centre of the firſt circle FSG, Q the centre of the ſecond FTH. Join and produce both ways the line PQ, and in it take QR in the ſame ratio to PQ as BC has to AB. But QR is to be taken towards that ſide of the point Q, that the order of the letters P, Q, R, may be the ſame as of the letters A, B, C; and about the centre R with the interval RF deſcribe a fourth circle FNc cutting the third circle FVI in c. Join Fc cutting the firſt circle in a, and the ſecond in b. Draw aG, bH, cI, and let the figure ABCfghi be made ſimilar to the figure abcFGHI; and the trapezium fghi will be that which was required to be deſcribed.

For let the two firſt circles FSG, FTH cut one the other in K; join PK, QK, RK, aK, bK, cK, and produce QP to L. The angles FaK, FbK, FcK at the circumferences are the halves of the angles FPK, FQK, FRK, at the centres, and therefore equal to LPK, LQK, LRK, the halves of thoſe angles. Wherefore the figure PQRK is equiangular

and similar to the figure abcK, and consequently ab is to bc as PQ to QR, that is, as AB to BC. But by construction, the angles fAg, fBh, fCi are equal to the angles FaG, FbH, FcI. And therefore the figure ABCfghi may be completed similar to the figure abcFGHI. Which done, a trapezium fghi will be constructed similar to the trapezium FGHI, and which by its angles f, g, h, i will touch the right lines ABC, AD, BD, CE. Q.E.F.

Cor. Hence a right line may be drawn whose parts intercepted in a given order, between four right lines given by position, shall have a given proportion among themselves. Let the angles FGH, GHI, be so far increased that the right lines FG, GH, HI, may lie *in directum;* and by constructing the problem in this case, a right line fghi will be drawn, whose parts fg, gh, hi, intercepted between the four right lines given by position, AB and AD, AD and BD, BD and CE, will be one to another as the lines FG, GH, HI, and will observe the same order among themselves. But the same thing may be more readily done in this manner.

Produce AB to K (Pl. 13, Fig. 2), and BD to L, so as BK may be to AB as HI to GH; and DL to BD as GI to FG; and join KL meeting the right line CE in i. Produce iL to M, so as LM may be to iL as GH to HI; then draw MQ parallel to LB, and meeting the right line AD in g, and join gi cutting AB, BD in f, h: I say, the thing is done.

For let Mg cut the right line AB in Q, and AD the right line KL in S, and draw AP parallel to BD, and meeting iL in P, and gM to Lh (gi to hi, Mi to Li, GI to HI, AK to BK) and AP to BL, will be in the same ratio. Cut DL in R, so as DL to RL may be in that same ratio; and because gS to gM, AS to AP, and DS to DL are proportional; therefore *(ex æquo)* as gS to Lh, so will AS be to BL, and DS to RL; and mixtly, BL — RL to Lh — BL, as AS — DS to gS — AS. That is, BR is to Bh as AD is to Ag, and therefore as BD to gQ. And alternately BR is to BD as Bh to gQ, or as fh to fg. But by construction the line BL was cut in D and R in the same ratio as the line FI in G and H; and therefore BR is to BD as FH to FG. Wherefore fh is to fg as FH to

FG. Since, therefore, gi to hi likewise is as Mi to Li, that is, as GI to HI, it is manifest that the lines FI, fi, are similarly cut in G and H, g and h. Q.E.F.

In the construction of this corollary, after the line LK is drawn cutting CE in i, we may produce iE to V, so as EV may be to Ei as FH to HI, and then draw Vf parallel to BD. It will come to the same, if about the centre i, with an interval IH, we describe a circle cutting BD in X, and produce iX to Y so as iY may be equal to IF, and then draw Yf parallel to BD.

Sir *Christopher Wren* and Dr. *Wallis* have long ago given other solutions of this problem.

PROPOSITION XXIX. PROBLEM XXI.

To describe a trajectory given in kind, that may be cut by four right lines given by position, into parts given in order, kind, and proportion.

Suppose a trajectory is to be described that may be similar to the curve line FGHI (Pl. 13, Fig. 3), and whose parts, similar and proportional to the parts FG, GH, HI of the other, may be intercepted between the right lines AB and AD, AD and BD, BD and CE given by position, viz. the first between the first pair of those lines, the second between the second, and the third between the third. Draw the right lines FG, GH, HI, FI; and (by lem. 27) describe a trapezium fghi that may be similar to the trapezium FGHI, and whose angles f, g, h, i, may touch the right lines given by position, AB, AD, BD, CE, severally according to their order. And then about this trapezium describe a trajectory, that trajectory will be similar to the curve line FGHI.

SCHOLIUM.

This problem may be likewise constructed in the following manner. Joining FG, GH, HI, FI (Pl. 13, Fig. 4), produce GF to V, and join FH, IG, and make the angles CAK, DAL equal to the angles FGH, VFH. Let AK, AL meet the right line BD in K and L, and thence draw KM, LN, of which let KM make the angle AKM equal to the angle GHI, and be itself to AK as HI is to GH; and let LN make the angle ALN equal to the angle FHI, and be itself to AL as

HI to FH. But AK, KM, AL, LN are to be drawn towards thofe fides of the lines AD, AK, AL, that the letters CAKMC, ALKA, DALND may be carried round in the fame order as the letters FGHIF; and draw MN meeting the right line CE in i. Make the angle iEP equal to the angle IGF, and let PE be to Ei as FG to GI; and through P draw PQf that may with the right line ADE contain an angle PQE equal to the angle FIG, and may meet the right line AB in f, and join fi. But PE and PQ are to be drawn towards thofe fides of the lines CE, PE, that the circular order of the letters PEiP and PEQP may be the fame as of the letters FGHIF; and if upon the line fi, in the fame order of letters, and fimilar to the trapezium FGHI, a trapezium fghi is conftructed, and a trajectory given in kind is circumfcribed about it, the problem will be folved.

So far concerning the finding of the orbits. It remains that we determine the motions of bodies in the orbits fo found.

SECTION VI.

How the motions are to be found in given orbits.

PROPOSITION XXX. PROBLEM XXII.

To find at any affigned time the place of a body moving in a given parabolic trajectory.

Let S (Pl. 14, Fig. 1) be the focus, and A the principal vertex of the parabola; and fuppofe $4AS \times M$ equal to the parabolic area to be cut off APS, which either was defcribed by the radius SP, fince the body's departure from the vertex, or is to be defcribed thereby before its arrival there. Now the quantity of that area to be cut off is known from the time which is proportional to it. Bifect AS in G, and erect the perpendicular GH equal to 3M, and a circle defcribed about the centre H, with the interval HS, will cut the parabola in the place P required. For letting fall PO perpendicular on the axis, and drawing PH, there will be $AG^2 + GH^2$ ($= HP^2 = \overline{AO - AG}|^2 + \overline{PO - GH}|^2$) $= AO^2 + PO^2 - 2GAO - 2GH + PO + AG^2 + GH^2$. Whence $2GH \times PO$ ($= AO^2 + PO^2 - 2GAO$) $= AO^2 + \frac{3}{4} PO^2$. For AO^2 write $AO \times \frac{PO^2}{4AS}$; then dividing all the terms by 3PO, and multiplying

them by 2AS, we ſhall have $\frac{4}{3}$GH × AS (= $\frac{1}{6}$AO × PO + $\frac{1}{2}$AS × PO = $\frac{AO + 3AS}{6}$ × PO = $\frac{4AO - 3SO}{6}$ × PO = to the area $\overline{APO - SPO)}$ = to the area APS. But GH was 3M, and therefore $\frac{4}{3}$GH × AS is 4AS × M. Wherefore the area cut off APS is equal to the area that was to be cut off 4AS × M. Q.E.D.

Cor. 1. Hence GH is to AS as the time in which the body deſcribed the arc AP to the time in which the body deſcribed the arc between the vertex A and the perpendicular erected from the focus S upon the axis.

Cor. 2. And ſuppoſing a circle ASP perpetually to paſs through the moving body P, the velocity of the point H is to the velocity which the body had in the vertex A as 3 to 8; and therefore in the ſame ratio is the line GH to the right line which the body, in the time of its moving from A to P, would deſcribe with that velocity which it had in the vertex A.

Cor. 3. Hence alſo, on the other hand, the time may be found in which the body has deſcribed any aſſigned arc AP. Join AP, and on its middle point erect a perpendicular meeting the right line GH in H.

LEMMA XXVIII.

There is no oval figure whoſe area, cut off by right lines at pleaſure, can be univerſally found by means of equations of any number of finite terms and dimenſions.

Suppoſe that within the oval any point is given, about which as a pole a right line is perpetually revolving with an uniform motion, while in that right line a moveable point going out from the pole moves always forward with a velocity proportional to the ſquare of that right line within the oval. By this motion that point will deſcribe a ſpiral with infinite circumgyrations. Now if a portion of the area of the oval cut off by that right line could be found by a finite equation, the diſtance of the point from the pole, which is proportional to this area, might be found by the ſame equation, and therefore all the points of the ſpiral might be found by a finite equation alſo; and therefore the interſection of a right line given in poſition with the ſpiral might alſo be found by a

finite equation. But every right line infinitely produced cuts a ſpiral in an infinite number of points; and the equation by which any one interſection of two lines is found at the ſame time exhibits all their interſections by as many roots, and therefore riſes to as many dimenſions as there are interſections. Becauſe two circles mutually cut one another in two points, one of thoſe interſections is not to be found but by an equation of two dimenſions, by which the other interſection may be alſo found. Becauſe there may be four interſections of two conic ſections, any one of them is not to be found univerſally, but by an equation of four dimenſions, by which they may be all found together. For if thoſe interſections are ſeverally ſought, becauſe the law and condition of all is the ſame, the calculus will be the ſame in every caſe, and therefore the concluſion always the ſame, which muſt therefore comprehend all thoſe interſections at once within itſelf, and exhibit them all indifferently. Hence it is that the interſections of the conic ſections with the curves of the third order, becauſe they may amount to ſix, come out together by equations of ſix dimenſions; and the interſections of two curves of the third order, becauſe they may amount to nine, come out together by equations of nine dimenſions. If this did not neceſſarily happen, we might reduce all ſolid to plane problems, and thoſe higher than ſolid to ſolid problems. But here I ſpeak of curves irreducible in power. For if the equation by which the curve is defined may be reduced to a lower power, the curve will not be one ſingle curve, but compoſed of two, or more, whoſe interſections may be ſeverally found by different calculuſſes. After the ſame manner the two interſections of right lines with the conic ſections come out always by equations of two dimenſions; the three interſections of right lines with the irreducible curves of the third order by equations of three dimenſions; the four interſections of right lines with the irreducible curves of the fourth order, by equations of four dimenſions; and ſo on *in infinitum*. Wherefore the innumerable interſections of a right line with a ſpiral, ſince this is but one ſimple curve, and not reducible to more curves, require equations infinite in number of dimenſions and

roots, by which they may be all exhibited together. For the law and calculus of all is the same. For if a perpendicular is let fall from the pole upon that interfecting right line, and that perpendicular together with the interfecting line revolves about the pole, the interfections of the fpiral will mutually pafs the one into the other; and that which was firft or neareft, after one revolution, will be the fecond; after two, the third; and fo on: nor will the equation in the mean time be changed but as the magnitudes of thofe quantities are changed, by which the pofition of the interfecting line is determined. Wherefore fince thofe quantities after every revolution return to their firft magnitudes, the equation will return to its firft form; and confequently one and the fame equation will exhibit all the interfections, and will therefore have an infinite number of roots, by which they may be all exhibited. And therefore the interfection of a right line with a fpiral cannot be univerfally found by any finite equation; and of confequence there is no oval figure whofe area, cut off by right lines at pleafure, can be univerfally exhibited by any fuch equation.

By the fame argument, if the interval of the pole and point by which the fpiral is defcribed is taken proportional to that part of the perimeter of the oval which is cut off, it may be proved that the length of the perimeter cannot be univerfally exhibited by any finite equation. But here I fpeak of ovals that are not touched by conjugate figures running out *in infinitum*.

Cor. Hence the area of an ellipfis, defcribed by a radius drawn from the focus to the moving body, is not to be found from the time given by a finite equation; and therefore cannot be determined by the defcription of curves geometrically rational. Thofe curves I call geometrically rational, all the points whereof may be determined by lengths that are definable by equations; that is, by the complicated ratios of lengths. Other curves (fuch as fpirals, quadratrixes, and cycloids) I call geometrically irrational. For the lengths which are or are not as number to number (according to the tenth book of elements) are arithmetically rational or irrational.

And therefore I cut off an area of an ellipsis proportional to the time in which it is described by a curve geometrically irrational, in the following manner.

PROPOSITION XXXI. PROBLEM XXIII.

To find the place of a body moving in a given elliptic trajectory at any assigned time.

Suppose A (Pl. 14, Fig. 2) to be the principal vertex, S the focus, and O the centre of the ellipsis APB; and let P be the place of the body to be found. Produce OA to G so as OG may be to OA as OA to OS. Erect the perpendicular GH; and about the centre O, with the interval OG, describe the circle GEF; and on the ruler GH, as a base, suppose the wheel GEF to move forwards, revolving about its axis, and in the mean time by its point A describing the cycloid ALI. Which done, take GK to the perimeter GEFG of the wheel, in the ratio of the time in which the body proceeding from A described the arc AP, to the time of a whole revolution in the ellipsis. Erect the perpendicular KL meeting the cycloid in L; then LP drawn parallel to KG will meet the ellipsis in P, the required place of the body.

For about the centre O with the interval OA describe the semi-circle AQB, and let LP, produced, if need be, meet the arc AQ in Q, and join SQ, OQ. Let OQ meet the arc EFG in F, and upon OQ let fall the perpendicular SR. The area APS is as the area AQS, that is, as the difference between the sector OQA and the triangle OQS, or as the difference of the rectangles $\frac{1}{2}OQ \times AQ$, and $\frac{1}{2}OQ \times SR$, that is, because $\frac{1}{2}OQ$ is given, as the difference between the arc AQ and the right line SR; and therefore (because of the equality of the given ratios SR to the sine of the arc AQ, OS to OA, OA to OG, AQ to GF; and by division, AQ — SR to GF — sine of the arc AQ) as GK, the difference between the arc GF and the sine of the arc AQ. Q.E.D.

SCHOLIUM.

But since the description of this curve is difficult, a solution by approximation will be preferable. First, then, let there be found a certain angle B which may be to an angle of 57,29578 degrees, which an arc equal to the radius subtends, as SH

(Pl. 14, Fig. 3), the distance of the foci, to AB, the diameter of the ellipsis. Secondly, a certain length L, which may be to the radius in the same ratio inversely. And these being found, the problem may be solved by the following analysis. By any construction (or even by conjecture), suppose we know P the place of the body near its true place p. Then letting fall on the axis of the ellipsis the ordinate PR from the proportion of the diameters of the ellipsis, the ordinate RQ of the circumscribed circle AQB will be given; which ordinate is the sine of the angle AOQ, supposing AO to be the radius, and also cuts the ellipsis in P. It will be sufficient if that angle is found by a rude calculus in numbers near the truth. Suppose we also know the angle proportional to the time, that is, which is to four right angles as the time in which the body described the arc Ap, to the time of one revolution in the ellipsis. Let this angle be N. Then take an angle D, which may be to the angle B as the sine of the angle AOQ to the radius; and an angle E which may be to the angle N — AOQ + D as the length L to the same length L diminished by the cosine of the angle AOQ, when that angle is less than a right angle, or increased thereby when greater. In the next place, take an angle F that may be to the angle B as the sine of the angle AOQ + E to the radius, and an angle G, that may be to the angle N — AOQ — E + F as the length L to the same length L diminished by the cosine of the angle AOQ + E, when that angle is less than a right angle, or increased thereby when greater. For the third time take an angle H, that may be to the angle B as the sine of the angle AOQ + E + G to the radius; and an angle I to the angle N — AOQ — E — G + H, as the length L is to the same length L diminished by the cosine of the angle AOQ + E + G, when that angle is less than a right angle, or increased thereby when greater. And so we may proceed *in infinitum*. Lastly, take the angle AOq equal to the angle AOQ + E + G + I +, &c. and from its cosine Or and the ordinate pr, which is to its sine qr as the lesser axis of the ellipsis to the greater, we shall have p the correct place of the body. When the angle N — AOQ + D happens to be negative, the sign + of the angle E must

be every where changed into —, and the ſign — into +. And the ſame thing is to be underſtood of the ſigns of the angles G and I, when the angles N — AOQ — E + F, and N — AOQ — E — G + H come out negative. But the infinite ſeries AOQ + E + G + I +, &c. converges ſo very faſt, that it will be ſcarcely ever needful to proceed beyond the ſecond term E. And the calculus is founded upon this theorem, that the area APS is as the difference between the arc AQ and the right line let fall from the focus S perpendicularly upon the radius OQ.

And by a calculus not unlike, the problem is ſolved in the hyperbola. Let its centre be O (Pl. 14, Fig. 4), its vertex A, its focus S, and aſymptote OK; and ſuppoſe the quantity of the area to be cut off is known, as being proportional to the time. Let that be A, and by conjecture ſuppoſe we know the poſition of a right line SP, that cuts off an area APS near the truth. Join OP, and from A and P to the aſymptote draw AI, PK parallel to the other aſymptote; and by the table of logarithms the area AIKP will be given, and equal thereto the area OPA, which ſubducted from the triangle OPS, will leave the area cut off APS. And by applying 2APS — 2A, or 2A — 2APS, the double difference of the area A that was to be cut off, and the area APS that is cut off, to the line SN that is let fall from the focus S, perpendicular upon the tangent TI, we ſhall have the length of the chord PQ. Which chord PQ is to be inſcribed between A and P, if the area APS that is cut off be greater than the area A that was to be cut off, but towards the contrary ſide of the point P, if otherwiſe: and the point Q will be the place of the body more accurately. And by repeating the computation the place may be found perpetually to greater and greater accuracy.

And by ſuch computations we have a general analytical reſolution of the problem. But the particular calculus that follows is better fitted for aſtronomical purpoſes. Suppoſing AO, OB, OD (Pl. 14, Fig. 5), to be the ſemi-axes of the ellipſis, and L its latus rectum, and D the difference betwixt the leſſer ſemi-axis OD, and $\frac{1}{2}$L the half of the latus rectum: let an angle Y be found, whoſe ſine may be to the radius as the rectan-

gle under that difference D, and AO + OD the half ſum of the axes to the ſquare of the greater axis AB. Find alſo an angle Z, whoſe ſine may be to the radius as the double rectangle under the diſtance of the foci SH and that difference D to triple the ſquare of half the greater ſemi-axis AO. Thoſe angles being once found, the place of the body may be thus determined. Take the angle T proportional to the time in which the arc BP was deſcribed, or equal to what is called the mean motion; and an angle V, the firſt equation of the mean motion to the angle Y, the greateſt firſt equation, as the ſine of double the angle T is to the radius; and an angle X, the ſecond equation, to the angle Z, the ſecond greateſt equation, as the cube of the ſine of the angle T is to the cube of the radius. Then take the angle BHP the mean motion equated equal to T + X + V, the ſum of the angles T, V, X, if the angle T is leſs than a right angle; or equal to T + X — V, the difference of the ſame, if that angle T is greater than one and leſs than two right angles; and if HP meets the ellipſis in P, draw SP, and it will cut off the area BSP nearly proportional to the time.

This practice ſeems to be expeditious enough, becauſe the angles V and X, taken in ſecond minutes, if you pleaſe, being very ſmall, it will be ſufficient to find two or three of their firſt figures. But it is likewiſe ſufficiently accurate to anſwer to the theory of the planets' motions. For even in the orbit of Mars, where the greateſt equation of the centre amounts to ten degrees, the error will ſcarcely exceed one ſecond. But when the angle of the mean motion equated BHP is found, the angle of the true motion BSP, and the diſtance SP, are readily had by the known methods.

And ſo far concerning the motion of bodies in curve lines. But it may alſo come to paſs that a moving body ſhall aſcend or deſcend in a right line; and I ſhall now go on to explain what belongs to ſuch kind of motions.

SECTION VII.

Concerning the rectilinear aſcent and deſcent of bodies.

PROPOSITION XXXII. PROBLEM XXIV.

Suppoſing that the centripetal force is reciprocally proportional to the ſquare of the diſtance of the places from the

centre; it is required to define the ſpaces which a body, falling directly, deſcribes in given times.

CASE 1. If the body does not fall perpendicularly, it will (by cor. 1, prop. 13) deſcribe ſome conic ſection whoſe focus is placed in the centre of force. Suppoſe that conic ſection to be ARPB (Pl. 15, Fig. 1), and its focus S. And, firſt, if the figure be an ellipſis, upon the greater axis thereof AB deſcribe the ſemi-circle ADB, and let the right line DPC paſs through the falling body, making right angles with the axis; and drawing DS, PS, the area ASD will be proportional to the area ASP, and therefore alſo to the time. The axis AB ſtill remaining the ſame, let the breadth of the ellipſis be perpetually diminiſhed, and the area ASD will always remain proportional to the time. Suppoſe that breadth to be diminiſhed *in infinitum;* and the orbit APB in that caſe coinciding with the axis AB, and the focus S with the extreme point of the axis B, the body will deſcend in the right line AC, and the area ABD will become proportional to the time. Wherefore the ſpace AC will be given which the body deſcribes in a given time by its perpendicular fall from the place A, if the area ABD is taken proportional to the time, and from the point D the right line DC is let fall perpendicularly on the right line AB. Q.E.I.

CASE 2. If the figure RPB is an hyperbola (Fig. 2), on the ſame principal diameter AB deſcribe the rectangular hyperbola BED; and becauſe the areas CSP, CBfP, SPfB, are ſeverally to the ſeveral areas CSD, CBED, SDEB, in the given ratio of the heights CP, CD, and the area SPfB is proportional to the time in which the body P will move through the arc PfB, the area SDEB will be alſo proportional to that time. Let the latus rectum of the hyperbola RPB be diminiſhed *in infinitum,* the latus tranſverſum remaining the ſame; and the arc PB will come to coincide with the right line CB, and the focus S with the vertex B, and the right line SD with the right line BD. And therefore the area BDEB will be proportional to the time in which the body C, by its perpendicular deſcent, deſcribes the line CB. Q.E.I.

CASE 3. And by the like argument, if the figure RPB is a parabola (Fig. 3), and to the same principal vertex B another parabola BED is described, that may always remain given while the former parabola in whose perimeter the body P moves, by having its latus rectum diminished and reduced to nothing, comes to coincide with the line CB, the parabolic segment BDEB will be proportional to the time in which that body P or C will descend to the centre S or B. Q.E.I.

PROPOSITION XXXIII. THEOREM IX.

The things above found being supposed, I say, that the velocity of a falling body in any place C *is to the velocity of a body, describing a circle about the centre* B *at the distance* BC, *in the subduplicate ratio of* AC, *the distance of the body from the remoter vertex* A *of the circle or rectangular hyperbola, to* $\frac{1}{2}$AB, *the principal semi-diameter of the figure.* (Pl. 15, Fig. 4.)

Let AB, the common diameter of both figures RPB, DEB, be bisected in O; and draw the right line PT that may touch the figure RPB in P, and likewise cut that common diameter AB (produced, if need be) in T; and let SY be perpendicular to this line, and BQ to this diameter, and suppose the latus rectum of the figure RPB to be L. From cor. 9, prop. 16, it is manifest that the velocity of a body, moving in the line RPB about the centre S, in any place P, is to the velocity of a body describing a circle about the same centre, at the distance SP, in the subduplicate ratio of the rectangle $\frac{1}{2}L \times SP$ to SY^2. For by the properties of the conic sections ACB is to CP^2 as 2AO to L, and therefore $\frac{2CP^2 \times AO}{ACB}$ is equal to L. Therefore those velocities are to each other in the subduplicate ratio of $\frac{CP^2 \times AO \times SP}{ACB}$ to SY^2. Moreover, by the properties of the conic sections, CO is to BO as BO to TO, and (by composition or division) as CB to BT. Whence (by division or composition) BO — or + CO will be to BO as CT to BT, that is, AC will be to AO as CP to BQ; and therefore $\frac{CP^2 \times AO \times SP}{ACB}$ is equal to $\frac{BQ^2 \times AC \times SP}{AO \times BC}$.

Now ſuppoſe CP, the breadth of the figure RPB, to be diminiſhed *in infinitum*, ſo as the point P may come to coincide with the point C, and the point S with the point B, and the line SP with the line BC, and the line SY with the line BQ; and the velocity of the body now deſcending perpendicularly in the line CB will be to the velocity of a body deſcribing a circle about the centre B, at the diſtance BC, in the ſubduplicate ratio of $\frac{BQ^2 \times AC \times SP}{AO \times BC}$ to SY^2, that is (neglecting the ratios of equality of SP to BC, and BQ^2 to SY^2), in the ſubduplicate ratio of AC to AO, or $\frac{1}{2}$AB. Q.E.D.

COR. 1. When the points B and S come to coincide, TC will become to TS as AC to AO.

COR. 2. A body revolving in any circle at a given diſtance from the centre, by its motion converted upwards, will aſcend to double its diſtance from the centre.

PROPOSITION XXXIV. THEOREM X.

If the figure BED *is a parabola, I ſay, that the velocity of a falling body in any place* C *is equal to the velocity by which a body may uniformly deſcribe a circle about the centre* B *at half the interval* BC. (Pl. 15, Fig. 5.)

For (by cor. 7, prop. 16) the velocity of a body deſcribing a parabola RPB about the centre S, in any place P, is equal to the velocity of a body uniformly deſcribing a circle about the ſame centre S at half the interval SP. Let the breadth CP of the parabola be diminiſhed *in infinitum*, ſo as the parabolic arc PfB may come to coincide with the right line CB, the centre S with the vertex B, and the interval SP with the interval BC, and the propoſition will be manifeſt. Q.E.D.

PROPOSITION XXXV. THEOREM XI.

The ſame things ſuppoſed, I ſay, that the area of the figure DES, *deſcribed by the indefinite radius* SD, *is equal to the area which a body with a radius equal to half the latus rectum of the figure* DES, *by uniformly revolving about the centre* S, *may deſcribe in the ſame time.* (Pl. 16, Fig. 1.)

For ſuppoſe a body C in the ſmalleſt moment of time deſcribes in falling the infinitely little line Cc, while another

body K, uniformly revolving about the centre S in the circle OKk, describes the arc Kk. Erect the perpendiculars CD, cd, meeting the figure DES in D, d. Join SD, Sd, SK, Sk, and draw Dd meeting the axis AS in T, and thereon let fall the perpendicular SY.

CASE 1. If the figure DES is a circle, or a rectangular hyperbola, bisect its transverse diameter AS in O, and SO will be half the latus rectum. And because TC is to TD as Cc to Dd, and TD to TS as CD to SY; *ex æquo* TC, will be to TS as CD × Cc to SY × Dd. But (by cor. 1, prop. 33) TC is to TS as AC to AO; to wit, if in the coalescence of the points D, d, the ultimate ratios of the lines are taken. Wherefore AC is to AO or SK as CD × Cc to SY × Dd. Farther, the velocity of the descending body in C is to the velocity of a body describing a circle about the centre S, at the interval SC, in the subduplicate ratio of AC to AO or SK (by prop. 33); and this velocity is to the velocity of a body describing the circle OKk in the subduplicate ratio of SK to SC (by cor. 6, prop. 4); and, *ex æquo,* the first velocity to the last, that is, the little line Cc to the arc Kk, in the subduplicate ratio of AC to SC, that is, in the ratio of AC to CD. Wherefore CD × Cc is equal to AC × Kk, and consequently AC to SK as AC × Kk to SY × Dd, and thence SK × Kk equal to SY × Dd, and $\frac{1}{2}$SK × Kk equal to $\frac{1}{2}$SY × Dd, that is, the area KSk equal to the area SDd. Therefore in every moment of time two equal particles, KSk and SDd, of areas are generated, which, if their magnitude is diminished, and their number increased *in infinitum,* obtain the ratio of equality, and consequently (by cor. lem. 4), the whole areas together generated are always equal. Q.E.D.

CASE 2. But if the figure DES (Fig. 2) is a parabola, we shall find, as above, CD × Cc to SY × Dd as TC to TS, that is, as 2 to 1; and that therefore $\frac{1}{4}$CD × Cc is equal to $\frac{1}{2}$SY × Dd. But the velocity of the falling body in C is equal to the velocity with which a circle may be uniformly described at the interval $\frac{1}{2}$SC (by prop. 34). And this velocity to the velocity with which a circle may be described with the radius SK, that is, the little line Cc to the arc Kk, is (by

cor. 5, prop. 4) in the fubduplicate ratio of SK to $\frac{1}{2}$SC; that is, in the ratio of SK to $\frac{1}{2}$CD. Wherefore $\frac{1}{2}$SK × Kk is equal to $\frac{1}{2}$CD × Cc, and therefore equal to $\frac{1}{2}$SY × Dd; that is, the area KSk is equal to the area SDd, as above. Q.E.D.

PROPOSITION XXXVI. PROBLEM XXV.

To determine the times of the defcent of a body falling from a given place A. (Pl. 16, Fig. 3.)

Upon the diameter AS, the diftance of the body from the centre at the beginning, defcribe the femi-circle ADS, as likewife the femi-circle OKH equal thereto, about the centre S. From any place C of the body erect the ordinate CD. Join SD, and make the fector OSK equal to the area ASD. It is evident (by prop. 35) that the body in falling will defcribe the fpace AC in the fame time in which another body, uniformly revolving about the centre S, may defcribe the arc OK. Q.E.F.

PROPOSITION XXXVII. PROBLEM XXVI.

To define the times of the afcent or defcent of a body projected upwards or downwards from a given place. (Pl. 16, Fig. 4.)

Suppofe the body to go off from the given place G, in the direction of the line GS, with any velocity. In the duplicate ratio of this velocity to the uniform velocity in a circle, with which the body may revolve about the centre S at the given interval SG, take GA to $\frac{1}{2}$AS. If that ratio is the fame as of the number 2 to 1, the point A is infinitely remote; in which cafe a parabola is to be defcribed with any latus rectum to the vertex S, and axis SG; as appears by prop. 34. But if that ratio is lefs or greater than the ratio of 2 to 1, in the former cafe a circle, in the latter a rectangular hyperbola, is to be defcribed on the diameter SA; as appears by prop. 33. Then about the centre S, with an interval equal to half the latus rectum, defcribe the circle HkK; and at the place G of the afcending or defcending body, and at any other place C, erect the perpendiculars GI, CD, meeting the conic fection or circle in I and D. Then joining SI, SD, let the fectors HSK, HSk be made equal to the fegments SEIS, SEDS, and (by prop. 35) the body G will defcribe the fpace

GC in the same time in which the body K may describe the arc Kk. Q.E.F.

PROPOSITION XXXVIII. THEOREM XII.

Supposing that the centripetal force is proportional to the altitude or distance of places from the centre, I say, that the times and velocities of falling bodies, and the spaces which they describe, are respectively proportional to the arcs, and the right and versed sines of the arcs. (Pl. 17, Fig. 1.)

Suppose the body to fall from any place A in the right line AS; and about the centre of force S, with the interval AS, describe the quadrant of a circle AE; and let CD be the right sine of any arc AD; and the body A will in the time AD in falling describe the space AC, and in the place C will acquire the velocity CD.

This is demonstrated the same way from prop. 10, as prop. 32 was demonstrated from prop. 11.

Cor. 1. Hence the times are equal in which one body falling from the place A arrives at the centre S, and another body revolving describes the quadrantal arc ADE.

Cor. 2. Wherefore all the times are equal in which bodies falling from whatsoever places arrive at the centre. For all the periodic times of revolving bodies are equal (by cor. 3, prop. 4).

PROPOSITION XXXIX. PROBLEM XXVII.

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures; it is required to find the velocity of a body, ascending or descending in a right line, in the several places through which it passes; as also the time in which it will arrive at any place: and vice versa.

Suppose the body E (Pl. 17, Fig. 2) to fall from any place A in the right line ADEC; and from its place E imagine a perpendicular EG always erected proportional to the centripetal force in that place tending to the centre C; and let BFG be a curve line, the locus of the point G. And in the beginning of the motion suppose EG to coincide with the perpendicular AB; and the velocity of the body in any place E will be as a right line whose power is the curvilinear area $\frac{1}{2}$ABGE. Q.E.I.

In EG take EM reciprocally proportional to a right line whofe power is the area $\frac{1}{2}$ABGE, and let VLM be a curve line wherein the point M is always placed, and to which the right line AB produced is an afymptote; and the time in which the body in falling defcribes the line AE, will be as the curvilinear area ABTVME. Q.E.I.

For in the right line AE let there be taken the very fmall line DE of a given length, and let DLF be the place of the line EMG, when the body was in D; and if the centripetal force be fuch, that a right line, whofe power is the area $\frac{1}{2}$ABGE, is as the velocity of the defcending body, the area itfelf will be as the fquare of that velocity; that is, if for the velocities in D and E we write V and V + I, the area ABFD will be as VV, and the area ABGE as VV + 2VI + II; and by divifion, the area DFGE as 2VI + II, and therefore $\frac{\text{DFGE}}{\text{DE}}$ will be as $\frac{2\text{VI} + \text{II}}{\text{DE}}$; that is, if we take the firft ratios of thofe quantities when juft nafcent, the length DF is as the quantity $\frac{2\text{VI}}{\text{DE}}$, and therefore alfo as half that quantity $\frac{\text{I} \times \text{V}}{\text{DE}}$. But the time in which the body in falling defcribes the very fmall line DE, is as that line directly and the velocity V inverfely; and the force will be as the increment I of the velocity directly and the time inverfely; and therefore if we take the firft ratios when thofe quantities are juft nafcent, as $\frac{\text{I} \times \text{V}}{\text{DE}}$, that is, as the length DF. Therefore a force proportional to DF or EG will caufe the body to defcend with a velocity that is as the right line whofe power is the area $\frac{1}{2}$ABGE. Q.E.D.

Moreover, fince the time in which a very fmall line DE of a given length may be defcribed is as the velocity inverfely, and therefore alfo inverfely as a right line whofe fquare is equal to the area ABFD; and fince the line DL, and by confequence the nafcent area DLME, will be as the fame right

line inversely, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas; that is (by cor. lem. 4), the whole time in which the line AE is described will be as the whole area ATVME. Q.E.D.

Cor 1. Let P be the place from whence a body ought to fall, so as that, when urged by any known uniform centripetal force (such as gravity is vulgarly supposed to be), it may acquire in the place D a velocity equal to the velocity which another body, falling by any force whatever, hath acquired in that place D. In the perpendicular DF let there be taken DR, which may be to DF as that uniform force to the other force in the place D. Complete the rectangle PDRQ, and cut off the area ABFD equal to that rectangle. Then A will be the place from whence the other body fell. For completing the rectangle DRSE, since the area ABFD is to the area DFGE as VV to 2VI, and therefore as $\frac{1}{2}$V to I, that is, as half the whole velocity to the increment of the velocity of the body falling by the unequable force; and in like manner the area PQRD to the area DRSE as half the whole velocity to the increment of the velocity of the body falling by the uniform force; and since those increments (by reason of the equality of the nascent times) are as the generating forces, that is, as the ordinates DF, DR, and consequently as the nascent areas DFGE, DRSE; therefore, *ex æquo*, the whole areas ABFD, PQRD will be to one another as the halves of the whole velocities; and therefore, because the velocities are equal, they become equal also.

Cor. 2. Whence if any body be projected either upwards or downwards with a given velocity from any place D, and there be given the law of centripetal force acting on it, its velocity will be found in any other place, as e, by erecting the ordinate eg, and taking that velocity to the velocity in the place D as a right line whose square is equal to the rectangle PQRD, either increased by the curvilinear area DFge, if the place e is below the place D, or diminished by the same area DFge, if it be higher, is to the right line whose square is equal to the rectangle PQRD alone.

COR. 3. The time is alfo known by erecting the ordinate em reciprocally proportional to the fquare root of PQRD + or — DFge, and taking the time in which the body has defcribed the line De to the time in which another body has fallen with an uniform force from P, and in falling arrived at D in the proportion of the curvilinear area DLme to the rectangle 2PD × DL. For the time in which a body falling with an uniform force hath defcribed the line PD, is to the time in which the fame body has defcribed the line PE in the fubduplicate ratio of PD to PE; that is (the very fmall line DE being juft nafcent), in the ratio of PD to PD + $\frac{1}{2}$DE, or 2PD to 2PD + DE, and, by divifion, to the time in which the body hath defcribed the fmall line DE, as 2PD to DE, and therefore as the rectangle 2PD × DL to the area DLME; and the time in which both the bodies defcribed the very fmall line DE is to the time in which the body moving unequably hath defcribed the line De as the area DLME to the area DLme; and, *ex æquo*, the firft mentioned of thefe times is to the laft as the rectangle 2PD × DL to the area DLme.

SECTION VIII.

Of the invention of orbits wherein bodies will revolve, being acted upon by any fort of centripetal force.

PROPOSITION XL. THEOREM XIII.

If a body, acted upon by any centripetal force, is any how moved, and another body afcends or defcends in a right line, and their velocities be equal in any one cafe of equal altitudes, their velocities will be alfo equal at all equal altitudes.

Let a body defcend from A (Pl. 17, Fig. 3) through D and E, to the centre C; and let another body move from V in the curve line VIKk. From the centre C, with any diftances, defcribe the concentric circles DI, EK, meeting the right line AC in D and E, and the curve VIK in I and K. Draw IC meeting KE in N, and on IK let fall the perpendicular NT; and let the interval DE or IN between the circumferences of the circles be very fmall; and imagine the bodies in D and I to have equal velocities. Then becaufe the diftances CD and CI are equal, the centripetal forces in

D and I will be alſo equal. Let thoſe forces be expreſſed by the equal lineolæ DE and IN; and let the force IN (by cor. 2 of the laws of motion) be reſolved into two others, NT and IT. Then the force NT acting in the direction of the line NT perpendicular to the path ITK of the body will not at all affect or change the velocity of the body in that path, but only draw it aſide from a rectilinear courſe, and make it deflect perpetually from the tangent of the orbit, and proceed in the curvilinear path ITKk. That whole force, therefore, will be ſpent in producing this effect; but the other force IT, acting in the direction of the courſe of the body, will be all employed in accelerating it, and in the leaſt given time will produce an acceleration proportional to itſelf. Therefore the accelerations of the bodies in D and I, produced in equal times, are as the lines DE, IT (if we take the firſt ratios of the naſcent lines DE, IN, IK, IT, NT); and in unequal times as thoſe lines and the times conjunctly. But the times in which DE and IK are deſcribed, are, by reaſon of the equal velocities (in D and I) as the ſpaces deſcribed DE and IK, and therefore the accelerations in the courſe of the bodies through the lines DE and IK are as DE and IT, and DE and IK conjunctly; that is, as the ſquare of DE to the rectangle IT into IK. But the rectangle IT × IK is equal to the ſquare of IN, that is, equal to the ſquare of DE; and therefore the accelerations generated in the paſſage of the bodies from D and I to E and K are equal. Therefore the velocities of the bodies in E and K are alſo equal: and by the ſame reaſoning they will always be found equal in any ſubſequent equal diſtances. Q.E.D.

By the ſame reaſoning, bodies of equal velocities and equal diſtances from the centre will be equally retarded in their aſcent to equal diſtances. Q.E.D.

Cor. 1. Therefore if a body either oſcillates by hanging to a ſtring, or by any poliſhed and perfectly ſmooth impediment is forced to move in a curve line; and another body aſcends or deſcends in a right line, and their velocities be equal at any one equal altitude, their velocities will be alſo equal at all other equal altitudes. For, by the ſtring of the pendulous body, or by the inpediment of a veſſel

perfectly smooth, the same thing will be effected as by the transverse force NT. The body is neither accelerated nor retarded by it, but only is obliged to quit its rectilinear course.

COR. 2. Suppose the quantity P to be the greatest distance from the centre to which a body can ascend, whether it be oscillating, or revolving in a trajectory, and so the same projected upwards from any point of a trajectory with the velocity it has in that point. Let the quantity A be the distance of the body from the centre in any other point of the orbit; and let the centripetal force be always as the power A^{n-1} of the quantity A, the index of which power $n-1$ is any number n diminished by unity. Then the velocity in every altitude A will be as $\sqrt{P^n - A^n}$, and therefore will be given. For by prop. 39, the velocity of a body ascending and descending in a right line is in that very ratio.

PROPOSITION XLI. PROBLEM XXVIII.

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures, it is required to find as well the trajectories in which bodies will move, as the times of their motions in the trajectories found.

Let any centripetal force tend to the centre C (Pl. 17, Fig. 4), and let it be required to find the trajectory VIKk. Let there be given the circle VR, described from the centre C with any interval CV; and from the same centre describe any other circles ID, KE cutting the trajectory in I and K, and the right line CV in D and E. Then draw the right line CNIX cutting the circles KE, VR in N and X, and the right line CKY meeting the circle VR in Y. Let the points I and K be indefinitely near; and let the body go on from V through I and K to k; and let the point A be the place from whence another body is to fall, so as in the place D to acquire a velocity equal to the velocity of the first body in I. And things remaining as in prop. 39, the lineola IK, described in the least given time, will be as the velocity, and therefore as the right line whose square is equal to the area $\frac{1}{2}$ABFD, and the triangle ICK proportional to the time will be given, and therefore KN will be reciprocally as the altitude IC; that is (if there be given any quantity Q,

and the altitude IC be called A), as $\frac{Q}{A}$. This quantity $\frac{Q}{A}$ call Z, and suppose the magnitude of Q to be such that in some case $\sqrt{ABFD}$ may be to Z as IK to KN, and then in all cases $\sqrt{ABFD}$ will be to Z as IK to KN, and ABFD to ZZ as IK^2 to KN^2, and by division ABFD — ZZ to ZZ as IN^2 to KN^2, and therefore $\sqrt{ABFD - ZZ}$ to Z, or $\frac{Q}{A}$ as IN to KN; and therefore $A \times KN$ will be equal to $\frac{Q \times IN}{\sqrt{ABFD - ZZ}}$. Therefore since $YX \times XC$ is to $A \times KN$ as CX^2 to AA, the rectangle $XY \times XC$ will be equal to $\frac{Q \times IN \times CX^2}{AA\sqrt{ABFD - ZZ}}$. Therefore in the perpendicular DF let there be taken continually Db, Dc equal to $\frac{Q}{2\sqrt{ABFD - ZZ}}$, $\frac{Q \times CX^2}{2AA\sqrt{ABFD - ZZ}}$ respectively, and let the curve lines ab, ac, the foci of the points b and c, be described: and from the point V let the perpendicular Va be erected to the line AC, cutting off the curvilinear areas VDba, VDca, and let the ordinates Ez, Ex, be erected also. Then because the rectangle $Db \times IN$ or DbzE is equal to half the rectangle $A \times KN$, or to the triangle ICK; and the rectangle $Dc \times IN$ or DcxE is equal to half the rectangle $YX \times XC$, or to the triangle XCY; that is, because the nascent particles DbzE, ICK of the areas VDba, VIC are always equal; and the nascent particles DcxE, XCY of the areas VDca, VCX are always equal; therefore the generated area VDba will be equal to the generated area VIC, and therefore proportional to the time; and the generated area VDca is equal to the generated sector VCX. If, therefore, any time be given during which the body has been moving from V, there will be also given the area porportional to it VDba; and thence will be given the altitude of the body CD or CI; and the area VDca, and the sector VCX equal thereto, together with its angle VCI. But the angle VCI, and the altitude CI being given, there is also given the place I, in which the body will be found at the end of that time. Q.E.I.

COR. 1. Hence the greateſt and leaſt altitudes of the bodies, that is, the apſides of the trajectories, may be found very readily. For the apſides are thoſe points in which a right line IC drawn through the centre falls perpendicularly upon the trajectory VIK; which comes to paſs when the right lines IK and NK become equal; that is, when the area ABFD is equal to ZZ.

COR. 2. So alſo the angle KIN, in which the trajectory at any place cuts the line IC, may be readily found by the given altitude IC of the body: to wit, by making the ſine of that angle to radius as KN to IK; that is, as Z to the ſquare root of the area ABFD.

COR. 3. If to the centre C (Pl. 17, Fig. 5), and the principal vertex V, there be deſcribed a conic ſection VRS; and from any point thereof, as R, there be drawn the tangent RT meeting the axis CV indefinitely produced in the point T; and then joining CR there be drawn the right line CP, equal to the abſciſſa CT, making an angle VCP proportional to the ſector VCR; and if a centripetal force, reciprocally proportional to the cubes of the diſtances of the places from the centre, tends to the centre C; and from the place V there ſets out a body with a juſt velocity in the direction of a line perpendicular to the right line CV; that body will proceed in a trajectory VPQ, which the point P will always touch; and therefore if the conic ſection VRS be an hyperbola, the body will deſcend to the centre; but if it be an ellipſis, it will aſcend perpetually, and go farther and farther off *in infinitum*. And, on the contrary, if a body endued with any velocity goes off from the place V, and according as it begins either to deſcend obliquely to the centre, or aſcends obliquely from it, the figure VRS be either an hyperbola or an ellipſis, the trajectory may be found by increaſing or diminiſhing the angle VCP in a given ratio. And the centripetal force becoming centrifugal, the body will aſcend obliquely in the trajectory VPQ, which is found by taking the angle VCP proportional to the elliptic ſector VRC, and the length CP equal to the length CT, as before. All theſe things follow from the foregoing propoſition, by the quadrature of a certain curve, the invention of which, as being eaſy enough, for brevity's ſake, I omit.

PROPOSITION XLII. PROBLEM XXIX.

The law of centripetal force being given, it is required to find the motion of a body ſetting out from a given place, with a given velocity, in the direction of a given right line.

Suppoſe the ſame things as in the three preceding propoſitions; and let the body go off from the place I (Pl. 17, Fig. 6) in the direction of the little line IK, with the ſame velocity as another body, by falling with an uniform centripetal force from the place P, may acquire in D; and let this uniform force be to the force with which the body is at firſt urged in I, as DR to DF. Let the body go on towards k; and about the centre C, with the interval Ck, deſcribe the circle ke, meeting the right line PD in e, and let there be erected the lines eg, ev, ew, ordinately applied to the curves BFg, abv, acw. From the given rectangle PDRQ and the given law of centripetal force, by which the firſt body is acted on, the curve line BFg is alſo given, by the conſtruction of prop. 27, and its cor. 1. Then from the given angle CIK is given the proportion of the naſcent lines IK, KN; and thence, by the conſtruction of prob. 28, there is given the quantity Q, with the curve lines abv, acw; and therefore, at the end of any time Dbve, there is given both the altitude of the body Ce or Ck, and the area Dcwe, with the ſector equal to it XCy, the angle ICk, and the place k, in which the body will then be found. Q.E.I.

We ſuppoſe in theſe propoſitions the centripetal force to vary in its receſs from the centre according to ſome law, which any one may imagine at pleaſure; but at equal diſtances from the centre to be every where the ſame.

I have hitherto conſidered the motions of bodies in immovable orbits. It remains now to add ſomething concerning their motions in orbits which revolve round the centres of force.

SECTION IX.*

Of the motion of bodies in moveable orbits; and of the motion of the apſides.

* On the ſubject of the ninth Section the reader may conſult an excellent "Memoir on the Inverſe Method of Central Forces," by that profoundly great and juſtly celebrated Mathematician, Dr. Dawſon, of Sedbergh, in Yorkſhire.

PROPOSITION XLIII. PROBLEM XXX.

It is required to make a body move in a trajectory that revolves about the centre of force in the same manner as another body in the same trajectory at rest.

In the orbit VPK (Pl. 18, Fig. 1), given by position, let the body P revolve, proceeding from V towards K. From the centre C let there be continually drawn Cp, equal to CP, making the angle VCp proportional to the angle VCP; and the area which the line Cp describes will be to the area VCP, which the line CP describes at the same time, as the velocity of the describing line Cp to the velocity of the describing line CP; that is, as the angle VCp to the angle VCP, therefore in a given ratio, and therefore proportional to the time. Since, then, the area described by the line Cp in an immovable plane is proportional to the time, it is manifest that a body, being acted upon by a just quantity of centripetal force, may revolve with the point p in the curve line which the same point p, by the method just now explained, may be madeto describe in an immovable plane. Make the angle VCu equal to the angle PCp, and the line Cu equal to CV, and the figure uCp equal to the figure VCP, and the body being always in the point p, will move in the perimeter of the revolving figure uCp, and will describe its (revolving) arc up in the same time that the other body P describes the similar and equal arc VP in the quiescent figure VPK. Find, then, by cor. 5, prop. 6, the centripetal force by which the body may be made to revolve in the curve line which the point p describes in an immovable plane, and the problem will be solved. Q.E.F.

PROPOSITION XLIV. THEOREM XIV.

The difference of the forces, by which two bodies may be made to move equally, one in a quiescent, the other in the same orbit revolving, is in a triplicate ratio of their common altitudes inversely.

Let the parts of the quiescent orbit VP, PK (Pl. 18, Fig. 2), be similar and equal to the parts of the revolving orbit up, pk; and let the distance of the points P and K be supposed of the utmost smallness. Let fall a perpendicular kr from the point k to the right line pC, and produce it to m, so that mr

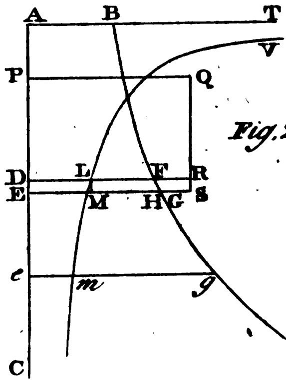


Fig. 2.

Fig. 4.

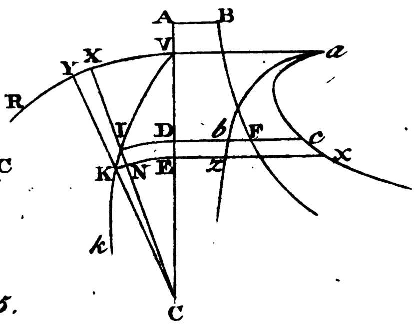
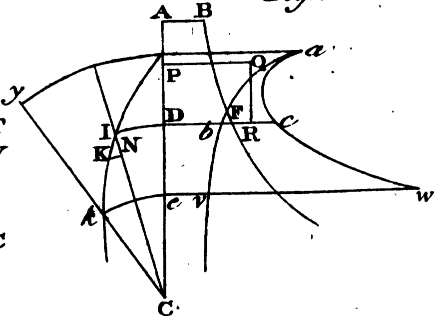


Fig. 6.



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may be to kr as the angle VCp to the angle VCP. Becaufe the altitudes of the bodies PC and pC, KC and kC, are always equal, it is manifeft that the increments or decrements of the lines PC and pC are always equal; and therefore if each of the feveral motions of the bodies in the places P and p be refolved into two (by cor. 2 of the laws of motion), one of which is directed towards the centre, or according to the lines PC, pC, and the other, tranfverfe to the former, hath a direction perpendicular to the lines PC and pC; the motions towards the centre will be equal, and the tranfverfe motion of the body p will be to the tranfverfe motion of the body P as the angular motion of the line pC to the angular motion of the line PC; that is, as the angle VCp to the angle VCP. Therefore, at the fame time that the body P, by both its motions, comes to the point K, the body p, having an equal motion towards the centre, will be equally moved from p towards C; and therefore that time being expired, it will be found fomewhere in the line mkr, which, paffing through the point k, is perpendicular to the line pC; and by its tranfverfe motion will acquire a diftance from the line pC, that will be to the diftance which the other body P acquires from the line PC as the tranfverfe motion of the body p to the tranfverfe motion of the other body P. Therefore fince kr is equal to the diftance which the body P acquires from the line PC, and mr is to kr as the angle VCp to the angle VCP, that is, as the tranfverfe motion of the body p to the tranfverfe motion of the body P, it is manifeft that the body p, at the expiration of that time, will be found in the place m. Thefe things will be fo, if the bodies p and P are equally moved in the directions of the lines pC and PC, and are therefore urged with equal forces in thofe directions. But if we take an angle pCn that is to the angle pCk as the angle VCp to the angle VCP, and nC be equal to kC, in that cafe the body p at the expiration of the time will really be in n; and is therefore urged with a greater force than the body P, if the angle nCp is greater than the angle kCp, that is, if the orbit upk move either *in confequentia*, or *in antecedentia*, with a celerity greater than the double of that with which the line CP moves *in con-*

ſequentia; and with a leſs force if the orbit moves ſlower *in antecedentia*. And the difference of the forces will be as the interval mn of the places through which the body would be carried by the action of that difference in that given ſpace of time. About the centre C with the interval Cn or Ck ſuppoſe a circle deſcribed cutting the lines mr, mn produced in s and t, and the rectangle mn × mt will be equal to the rectangle mk × ms, and therefore mn will be equal to $\frac{mk \times ms}{mt}$. But ſince the triangles pCk, pCn, in a given time, are of a given magnitude, kr and mr, and their difference mk, and their ſum ms, are reciprocally as the altitude pC, and therefore the rectangle mk × ms is reciprocally as the ſquare of the altitude pC. But, moreover, mt is directly as $\frac{1}{2}$mt, that is, as the altitude pC. Theſe are the firſt ratios of the naſcent lines; and hence $\frac{mk \times ms}{mt}$, that is, the naſcent lineolæ mn, and the difference of the forces proportional thereto, are reciprocally as the cube of the altitude pC. Q.E.D.

Cor. 1. Hence the difference of the forces in the places P and p, or K and k, is to the force with which a body may revolve with a circular motion from R to K, in the ſame time that the body P in an immovable orb deſcribes the arc PK, as the naſcent line mn to the verſed ſine of the naſcent arc RK, that is, as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$, or as mk × ms to the ſquare of rk; that is, if we take given quantities F and G in the ſame ratio to one another as the angle VCP bears to the angle VCp, as GG — FF to FF. And, therefore, if from the centre C, with any diſtance CP or Cp, there be deſcribed a circular ſector equal to the whole area VPC, which the body revolving in an immovable orbit has by a radius drawn to the centre deſcribed in any certain time, the difference of the forces, with which the body P revolves in an immovable orbit, and the body p in a moveable orbit, will be to the centripetal force, with which another body by a radius drawn to the centre can uniformly deſcribe that ſector in the ſame time

as the area VPC is described, as GG — FF to FF. For that sector and the area pCk are to one another as the times in which they are described.

COR. 2. If the orbit VPK be an ellipsis, having its focus C, and its highest apsis V, and we suppose the ellipsis upk similar and equal to it, so that pC may be always equal to PC, and the angle VCp be to the angle VCP in the given ratio of G to F; and for the altitude PC or pC we put A, and 2R for the latus rectum of the ellipsis, the force with which a body may be made to revolve in a moveable ellipsis will be as $\frac{FF}{AA}$ + $\frac{RGG - RFF}{A^3}$, and *vice versa*. Let the force with which a body may revolve in an immovable ellipsis be expressed by the quantity $\frac{FF}{AA}$, and the force in V will be $\frac{FF}{CV^2}$. But the force with which a body may revolve in a circle at the distance CV, with the same velocity as a body revolving in an ellipsis has in V, is to the force with which a body revolving in an ellipsis is acted upon in the apsis V, as half the latus rectum of the ellipsis to the semi-diameter CV of the circle, and therefore is as $\frac{RFF}{CV^3}$; and the force which is to this, as GG — FF to FF, is as $\frac{RGG - RFF}{CV^3}$: and this force (by cor. 1 of this prop.) is the difference of the forces in V, with which the body P revolves in the immovable ellipsis VPK, and the body p in the moveable ellipsis upk. Therefore since by this prop. that difference at any other altitude A is to itself at the altitude CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference in every altitude A will be as $\frac{RGG - RFF}{A^3}$. Therefore to the force $\frac{FF}{AA}$, by which the body may revolve in an immovable ellipsis VPK, add the excess $\frac{RGG - RFF}{A^3}$, and the sum will be the whole force $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$ by which a body may revolve in the same time in the moveable ellipsis upk.

Cor. 3. In the same manner it will be found, that, if the immovable orbit VPK be an ellipsis having its centre in the centre of the forces C, and there be supposed a moveable ellipsis upk, similar, equal, and concentrical to it; and 2R be the principal latus rectum of that ellipsis, and 2T the latus transversum, or greater axis; and the angle VCp be continually to the angle VCP as G to F; the forces with which bodies may revolve in the immovable and moveable ellipsis, in equal times, will be as $\frac{FFA}{T^3}$ and $\frac{FFA}{T^3} + \frac{RGG - RFF}{A^3}$ respectively.

Cor. 4. And universally, if the greatest altitude CV of the body be called T, and the radius of the curvature which the orbit VPK has in V, that is, the radius of a circle equally curve, be called R, and the centripetal force with which a body may revolve in any immovable trajectory VPK at the place V be called $\frac{VFF}{TT}$, and in other places P be indefinitely styled X; and the altitude CP be called A, and G be taken to F in the given ratio of the angle VCp to the angle VCP; the centripetal force with which the same body will perform the same motions in the same time in the same trajectory upk revolving with a circular motion, will be as the sum of the forces X + $\frac{VRGG - VRFF}{A^3}$.

Cor. 5. Therefore the motion of a body in an immovable orbit being given, its angular motion round the centre of the forces may be increased or diminished in a given ratio; and thence new immovable orbits may be found in which bodies may revolve with new centripetal forces.

Cor. 6. Therefore if there be erected (Pl. 13, Fig. 3) the line VP of an indeterminate length, perpendicular to the line CV given by position, and CP be drawn, and Cp equal to it, making the angle VCp having a given ratio to the angle VCP, the force with which a body may revolve in the curve line Vpk, which the point p is continually describing, will be reciprocally as the cube of the altitude Cp. For the body P, by its *vis inertiæ* alone, no other force impelling it, will proceed

uniformly in the right line VP. Add, then, a force tending to the centre C reciprocally as the cube of the altitude CP or Cp, and (by what was just demonstrated) the body will deflect from the rectilinear motion into the curve line Vpk. But this curve Vpk is the same with the curve VPQ found in cor. 3, prop. 41, in which, I said, bodies attracted with such forces would ascend obliquely.

PROPOSITION XLV. PROBLEM XXXI.

To find the motion of the apsides in orbits approaching very near to circles.

This problem is solved arithmetically by reducing the orbit, which a body revolving in a moveable ellipsis (as in cor. 2 and 3 of the above prop.) describes in an immovable plane, to the figure of the orbit whose apsides are required; and then seeking the apsides of the orbit which that body describes in an immovable plane. But orbits acquire the same figure, if the centripetal forces with which they are described, compared between themselves, are made proportional at equal altitudes. Let the point V be the highest apsis, and write T for the greatest altitude CV, A for any other altitude CP or Cp, and X for the difference of the altitudes CV — CP; and the force with which a body moves in an ellipsis revolving about its focus C (as in cor. 2), and which in cor. 2 was as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, that is, as $\frac{FFA + RGG - RFF}{A^3}$, by substituting T—X for A, will become as $\frac{RGG - RFF + TFF - FFX}{A^3}$. In like manner any other centripetal force is to be reduced to a fraction whose denominator is A^3, and the numerators are to be made analogous by collating together the homologous terms. This will be made plainer by examples.

Example 1. Let us suppose the centripetal force to be uniform, and therefore as $\frac{A^3}{A^3}$, or, writing T — X for A in the numerator, as $\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$.

$\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$. Then collating together the correspondent terms of the numerators, that is, thoſe that conſiſt of given quantities with thoſe of given quantities; and thoſe of quantities not given with thoſe of quantities not given, it will become $RGG - RFF + TFF$ to T^3 as $- FFX$ to $- 3TTX + 3TXX - X^3$, or as $- FF$ to $- 3TT + 3TX - XX$. Now ſince the orbit is ſuppoſed extremely near to a circle, let it coincide with a circle; and becauſe in that caſe R and T become equal, and X is infinitely diminiſhed, the laſt ratios will be, as RGG to T^3, ſo $- FF$ to $- 3TT$, or as GG to TT, ſo FF to $3TT$; and again, as GG to FF, ſo TT to $3TT$, that is, as 1 to 3; and therefore G is to F, that is, the angle VCp to the angle VCP, as 1 to $\sqrt{3}$. Therefore ſince the body, in an immovable ellipſis, in deſcending from the upper to the lower apſis, deſcribes an angle, if I may ſo ſpeak, of 180 deg., the other body in a moveable ellipſis, and therefore in the immovable orbit we are treating of, will, in its deſcent from the upper to the lower apſis, deſcribe an angle VCp of $\frac{180}{\sqrt{3}}$ deg. And this comes to paſs by reaſon of the likeneſs of this orbit which a body acted upon by an uniform centripetal force deſcribes, and of that orbit which a body performing its circuits in a revolving ellipſis will deſcribe in a quieſcent plane. By this collation of the terms, theſe orbits are made ſimilar; not univerſally, indeed, but then only when they approach very near to a circular figure. A body, therefore, revolving with an uniform centripetal force in an orbit nearly circular, will always deſcribe an angle of $\frac{180}{\sqrt{3}}$ deg., or 103 deg., 55 m., 23 ſec., at the centre; moving from the upper apſis to the lower apſis when it has once deſcribed that angle, and thence returning to the upper apſis when it has deſcribed that angle again; and ſo on *in infinitum*.

EXAM. 2. Suppoſe the centripetal force to be as any power of the altitude A, as, for example, A^{n-3}, or $\frac{A^n}{A^3}$; where $n - 3$ and n ſignify any indices of powers whatever, whether inte-

gers or fractions, rational or surd, affirmative or negative. That numerator A^n or $\overline{T - X}|^n$ being reduced to an indeterminate series by my method of converging series, will become $T^n - nXT^{n-1} + \frac{nn - n}{2} XXT^{n-2}$, &c. And conferring these terms with the terms of the other numerator RGG — RFF + TFF — FFX, it becomes as RGG — RFF + TFF to T^n, so — FF to $- nT^{n-1} + \frac{nn - n}{2} XT^{n-2}$, &c. And taking the last ratios where the orbits approach to circles, it becomes as RGG to T^n, so — FF to $- nT^{n-1}$, or as GG to T^{n-1}, so FF to nT^{n-1}; and again, GG to FF, so T^{n-1} to nT^{n-1}, that is, as 1 to n; and therefore G is to F, that is, the angle VCp to the angle VCP, as 1 to $\surd n$. Therefore since the angle VCP, described in the descent of the body from the upper apsis to the lower apsis in an ellipsis, is of 180 deg., the angle VCp, described in the descent of the body from the upper apsis to the lower apsis in an orbit nearly circular which a body describes with a centripetal force proportional to the power A^{n-3}, will be equal to an angle of $\frac{180}{\surd n}$ deg., and this angle being repeated, the body will return from the lower to the upper apsis, and so on *in infinitum.* As if the centripetal force be as the distance of the body from the centre, that is, as A, or $\frac{A^4}{A^3}$, n will be equal to 4, and $\surd n$ equal to 2; and therefore the angle between the upper and the lower apsis will be equal to $\frac{180}{2}$ deg., or 90 deg. Therefore the body having performed a fourth part of one revolution, will arrive at the lower apsis, and having performed another fourth part, will arrive at the upper apsis, and so on by turns *in infinitum.* This appears also from prop. 10. For a body acted on by this centripetal force will revolve in an immovable ellipsis, whose centre is the centre of force. If the centripetal force is reciprocally as the distance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n will be equal to 2; and therefore the angle between the up-

per and lower apsis will be $\frac{180}{\sqrt{2}}$ deg., or 127 deg., 16 min., 45 sec.; and therefore a body revolving with such a force, will, by a perpetual repetition of this angle, move alternately from the upper to the lower and from the lower to the upper apsis for ever. So, also, if the centripetal force be reciprocally as the biquadrate root of the eleventh power of the altitude, that is, reciprocally as $A^{\frac{11}{4}}$, and, therefore, directly as $\frac{1}{A^{\frac{11}{4}}}$, or as $\frac{A^{\frac{1}{4}}}{A^3}$, n will be equal to $\frac{1}{4}$, and $\frac{180}{\sqrt{n}}$ deg. will be equal to 360 deg.; and therefore the body parting from the upper apsis, and from thence perpetually descending, will arrive at the lower apsis when it has completed one entire revolution; and thence ascending perpetually, when it has completed another entire revolution, it will arrive again at the upper apsis; and so alternately for ever.

EXAM. 3. Taking m and n for any indices of the powers of the altitude, and b and c for any given numbers, suppose the centripetal force to be as $\frac{bA^m + cA^n}{A^3}$, that is, as $\frac{b \text{ into } \overline{T-X}|^m \; c \text{ into } \overline{T-X}|^n}{A^3}$ or (by the method of converging series above-mentioned) as $\frac{bT^m + cT^n - mbXT^{m-1} \; ncXT^{n-1}}{A^3}$ $+ \frac{mm-m}{2} bXXT^{m-2} + \frac{nn-n}{2} cXXT^{n-2}$, &c. and comparing the terms of the numerators, there will arise $RGG - RFF + TFF$ to $bT^m + cT^n$ as $-FF$ to $-mbT^{m-1} - ncT^{n-1} + \frac{mm-m}{2} bXT^{m-2} + \frac{nn-n}{2} cXT^{n-2}$, &c. And taking the last ratios that arise when the orbits come to a circular form, there will come forth GG to $bT^{m-1} + cT^{n-1}$ as FF to $mbT^{m-1} + ncT^{n-1}$; and again, GG to FF as $bT^{m-1} + cT^{n-1}$ to $mbT^{m-1} + ncT^{n-1}$. This proportion, by expressing the greatest altitude CV or T arithmetically by unity, becomes, GG to FF as b + c to mb + nc, and therefore as 1

to $\frac{mb + nc}{b + c}$. Whence G becomes to F, that is, the angle VCp to the angle VCP, as 1 to $\sqrt{\frac{mb + nc}{b + c}}$. And therefore since the angle VCP between the upper and the lower apsis, in an immovable ellipsis, is of 180 deg., the angle VCp between the same apsides in an orbit which a body describes with a centripetal force, that is, as $\frac{bA^m + cA^n}{A^3}$, will be equal to an angle of 180 $\sqrt{\frac{b + c}{mb + nc}}$ deg. And by the same reasoning, if the centripetal force be as $\frac{bA^m - cA^n}{A^3}$, the angle between the apsides will be found equal to 180 $\sqrt{\frac{b - c}{mb - nc}}$ deg. After the same manner the problem is solved in more difficult cases. The quantity to which the centripetal force is proportional must always be resolved into a converging series whose denominator is A^3. Then the given part of the numerator arising from that operation is to be supposed in the same ratio to that part of it which is not given, as the given part of this numerator RGG — RFF + TFF — FFX is to that part of the same numerator which is not given. And taking away the superfluous quantities, and writing unity for T, the proportion of G to F is obtained.

Cor. 1. Hence if the centripetal force be as any power of the altitude, that power may be found from the motion of the apsides; and so contrariwise. That is, if the whole angular motion, with which the body returns to the same apsis, be to the angular motion of one revolution, or 360 deg., as any number as m to another as n, and the altitude called A; the force will be as the power $A^{\frac{nn}{mm} - 3}$ of the altitude A; the index of which power is $\frac{nn}{mm} - 3$. This appears by the second example. Hence it is plain that the force in its recess from the centre cannot decrease in a greater than a triplicate ratio of the altitude. A body revolving with such a force, and

parting from the apsis, if it once begins to descend, can never arrive at the lower apsis or least altitude, but will descend to the centre, describing the curve line treated of in cor. 3, prop. 41. But if it should, at its parting from the lower apsis, begin to ascend never so little, it will ascend *in infinitum*, and never come to the upper apsis; but will describe the curve line spoken of in the same cor., and cor. 6, prop. 44. So that where the force in its recess from the centre decreases in a greater than a triplicate ratio of the altitude, the body, at its parting from the apsis, will either descend to the centre, or ascend *in infinitum*, according as it descends or ascends at the beginning of its motion. But if the force in its recess from the centre either decreases in a less than a triplicate ratio of the altitude, or increases in any ratio of the altitude whatsoever, the body will never descend to the centre, but will at some time arrive at the lower apsis; and, on the contrary, if the body alternately ascending and descending from one apsis to another never comes to the centre, then either the force increases in the recess from the centre, or it decreases in a less than a triplicate ratio of the altitude; and the sooner the body returns from one apsis to another, the farther is the ratio of the forces from the triplicate ratio. As if the body should return to and from the upper apsis by an alternate descent and ascent in 8 revolutions, or in 4, or 2, or $1\frac{1}{2}$; that is, if m should be to n as 8, or 4, or 2, or $1\frac{1}{2}$ to 1, and therefore $\frac{nn}{mm} - 3$ be $\frac{1}{64} - 3$, or $\frac{1}{16} - 3$, or $\frac{1}{4} - 3$, or $\frac{4}{9} - 3$; then the force will be as $A^{\frac{1}{64} - 3}$, or $A^{\frac{1}{16} - 3}$, or $A^{\frac{1}{4} - 3}$, or $A^{\frac{4}{9} - 3}$; that is, it will be reciprocally as $A^{3 - \frac{1}{64}}$, or $A^{3 - \frac{1}{16}}$, or $A^{3 - \frac{1}{4}}$, or $A^{3 - \frac{4}{9}}$. If the body after each revolution returns to the same apsis, and the apsis remains unmoved, then m will be to n as 1 to 1, and therefore $A^{\frac{nn}{mm} - 3}$ will be equal to A^{-2}, or $\frac{1}{AA}$; and therefore the decrease of the forces will be in a duplicate ratio of the altitude; as was demonstrated above. If the body in three fourth parts, or two thirds, or one third, or one fourth part of an entire

revolution, return to the same apsis; m will be to n as $\frac{1}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and therefore $A^{\frac{nn}{mm}-3}$ is equal to $A^{\frac{1}{16}-3}$, or $A^{\frac{9}{4}-3}$, or A^{9-3}, or A^{16-3}; and therefore the force is either reciprocally as $A^{\frac{47}{16}}$ or $A^{\frac{3}{4}}$, or directly as A^{6} or A^{13}. Lastly, if the body in its progress from the upper apsis to the same upper apsis again, goes over one entire revolution and three deg. more, and therefore that apsis in each revolution of the body moves three deg. *in consequentia*; then m will be to n as 363 deg. to 360 deg. or as 121 to 120, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to $A^{-\frac{29523}{14641}}$, and therefore the centripetal force will be reciprocally as $A^{\frac{29523}{14641}}$, or reciprocally as $A^{2\frac{49}{243}}$ very nearly. Therefore the centripetal force decreases in a ratio something greater than the duplicate; but approaching $59\frac{3}{4}$ times nearer to the duplicate than the triplicate.

COR. 2. Hence also if a body, urged by a centripetal force which is reciprocally as the square of the altitude, revolves in an ellipsis whose focus is in the centre of the forces; and a new and foreign force should be added to or subducted from this centripetal force, the motion of the apsides arising from that foreign force may (by the third example) be known; and so on the contrary. As if the force with which the body revolves in the ellipsis be as $\frac{1}{AA}$; and the foreign force subducted as cA, and therefore the remaining force as $\frac{A - cA^4}{A^3}$; then (by the third exam.) b will be equal to 1, m equal to 1, and n equal to 4; and therefore the angle of revolution between the apsides is equal to $180\sqrt{\frac{1-c}{1-4c}}$ deg. Suppose that foreign force to be 357.45 parts less than the other force with which the body revolves in the ellipsis; that is, c to be $\frac{100}{35745}$; A or T being equal to 1; and then $180\sqrt{\frac{1-c}{1-4c}}$ will be 180

$\sqrt{\frac{35545}{35245}}$ or 180.7623, that is, 180 deg., 45 min., 44 sec. Therefore the body, parting from the upper apsis, will arrive at the lower apsis with an angular motion of 180 deg., 45 min., 44 sec. and this angular motion being repeated will return to the upper apsis; and therefore the upper apsis in each revolution will go forward 1 deg., 31 m., 28 sec. The apsis of the moon is about twice as swift.

So much for the motion of bodies in orbits whose planes pass through the centre of force. It now remains to determine those motions in eccentrical planes. For those authors who treat of the motion of heavy bodies used to consider the ascent and descent of such bodies, not only in a perpendicular direction, but at all degrees of obliquity upon any given planes; and for the same reason we are to consider in this place the motions of bodies tending to centres by means of any forces whatsoever, when those bodies move in eccentrical planes. These planes are supposed to be perfectly smooth and polished, so as not to retard the motion of the bodies in the least. Moreover, in these demonstrations, instead of the planes upon which those bodies roll or slide, and which are therefore tangent planes to the bodies, I shall use planes parallel to them, in which the centres of the bodies move, and by that motion describe orbits. And by the same method I afterwards determine the motions of bodies performed in curve superficies.

SECTION X.

Of the motion of bodies in given superficies, and of the reciprocal motion of funependulous bodies.

PROPOSITION XLVI. PROBLEM XXXII.

Any kind of centripetal force being supposed, and the centre of force, and any plane whatsoever in which the body revolves, being given, and the quadratures of curvilinear figures being allowed; it is required to determine the motion of a body going off from a given place, with a given velocity, in the direction of a given right line in that plane.

Let S (Pl. 18, Fig. 4) be the centre of force, SC the least distance of that centre from the given plane, P a body issuing from the place P in the direction of the right line PZ, Q the

ſame body revolving in its trajectory, and PQR the trajectory itſelf which is required to be found, deſcribed in that given plane. Join CQ, QS, and if in QS we take SV proportional to the centripetal force with which the body is attracted towards the centre S, and draw VT parallel to CQ, and meeting SC in T; then will the force SV be reſolved into two (by cor. 2, of the laws of motion), the force ST, and the force TV; of which ST attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane. But the action of the other force TV, coinciding with the poſition of the plane itſelf, attracts the body directly towards the given point C in that plane; and therefore cauſes the body to move in this plane in the ſame manner as if the force ST were taken away, and the body were to revolve in free ſpace about the centre C by means of the force TV alone. But there being given the centripetal force TV with which the body Q revolves in free ſpace about the given centre C, there is given (by prop. 42) the trajectory PQR which the body deſcribes; the place Q, in which the body will be found at any given time; and, laſtly, the velocity of the body in that place Q. And ſo *è contra*. Q.E.I.

PROPOSITION XLVII. THEOREM XV.

Suppoſing the centripetal force to be proportional to the diſtance of the body from the centre; all bodies revolving in any planes whatſoever will deſcribe ellipſes, and complete their revolutions in equal times; and thoſe which move in right lines, running backwards and forwards alternately, will complete their ſeveral periods of going and returning in the ſame times.

For letting all things ſtand as in the foregoing propoſition, the force SV, with which the body Q revolving in any plane PQR is attracted towards the centre S, is as the diſtance SQ; and therefore becauſe SV and SQ, TV and CQ are proportional, the force TV with which the body is attracted towards the given point C in the plane of the orbit is as the diſtance CQ. Therefore the forces with which bodies found in the plane PQR are attracted towards the point C, are in proportion to the diſtances equal to the forces with which the ſame

bodies are attracted every way towards the centre S; and therefore the bodies will move in the fame times, and in the fame figures, in any plane PQR about the point C, as they would do in free fpaces about the centre S; and therefore (by cor. 2, prop. 10, and cor. 2, prop. 38) they will in equal times either defcribe ellipfes in that plane about the centre C, or move to and fro in right lines paffing through the centre C in that plane; completing the fame periods of time in all cafes. Q.E.D.

SCHOLIUM.

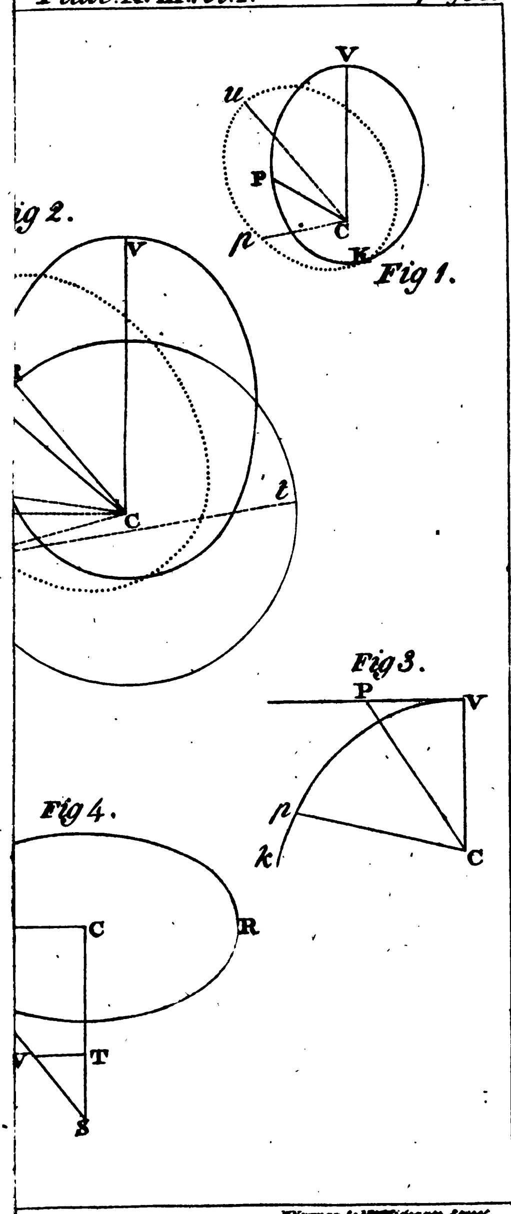
The afcent and defcent of bodies in curve fuperficies has a near relation to thefe motions we have been fpeaking of. Imagine curve lines to be defcribed on any plane, and to revolve about any given axes paffing through the centre of force, and by that revolution to defcribe curve fuperficies; and that the bodies move in fuch fort that their centres may be always found in thofe fuperficies. If thofe bodies reciprocate to and fro with an oblique afcent and defcent, their motions will be performed in planes paffing through the axis, and therefore in the curve lines, by whofe revolution thofe curve fuperficies were generated. In thofe cafes, therefore, it will be fufficient to confider the motion in thofe curve lines.

PROPOSITION XLVIII. THEOREM XVI.

If a wheel ftands upon the outfide of a globe at right angles thereto, and revolving about its own axis goes forward in a great circle, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath defcribed fince the time that it touched the globe (which curvilinear path we may call the cycloid or epicycloid), will be to double the verfed fine of half the arc which fince that time has touched the globe in paffing over it, as the fum of the diameters of the globe and the wheel to the femi-diameter of the globe.

PROPOSITION XLIX. THEOREM XVII.

If a wheel ftand upon the infide of a concave globe at right angles thereto, and revolving about its own axis go forward in one of the great circles of the globe, the length of the curvilinear path which any point, given in the perimeter of the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wheel, hath described since it touched the globe, will be to the double of the versed sine of half the arc which in all that time has touched the globe in passing over it, as the difference of the diameters of the globe and the wheel to the semi-diameter of the globe.

Let ABL (Pl. 19, Fig. 1, 2,) be the globe, C its centre, BPV the wheel insisting thereon, E the centre of the wheel, B the point of contact, and P the given point in the perimeter of the wheel. Imagine this wheel to proceed in the great circle ABL from A through B towards L, and in its progress to revolve in such a manner that the arcs AB, PB may be always equal the one to the other, and the given point P in the perimeter of the wheel may describe in the mean time the curvilinear path AP. Let AP be the whole curvilinear path described since the wheel touched the globe in A, and the length of this path AP will be to twice the versed sine of the arc ½PB as 2CE to CB. For let the right line CE (produced if need be) meet the wheel in V, and join CP, BP, EP, VP; produce CP, and let fall thereon the perpendicular VF. Let PH, VH, meeting in H, touch the circle in P and V, and let PH cut VF in G, and to VP let fall the perpendiculars GI, HK. From the centre C with any interval let there be described the circle nom, cutting the right line CP in n, the perimeter of the wheel BP in o, and the curvilinear path AP in m; and from the centre V with the interval Vo let there be described a circle cutting VP produced in q.

Because the wheel in its progress always revolves about the point of contact B, it is manifest that the right line BP is perpendicular to that curve line AP which the point P of the wheel describes, and therefore that the right line VP will touch this curve in the point P. Let the radius of the circle nom be gradually increased or diminished so that at last it become equal to the distance CP; and by reason of the similitude of the evanescent figure Pnomq, and the figure PFGVI, the ultimate ratio of the evanescent lineolæ Pm, Pn, Po, Pq, that is, the ratio of the momentary mutations of the curve AP, the right line CP, the circular arc BP, and the right line VP, will be the same as of the lines PV, PF, PG, PI,

respectively. But since VF is perpendicular to CF, and VH to CV, and therefore the angles HVG, VCF equal; and the angle VHG (because the angles of the quadrilateral figure HVEP are right in V and P) is equal to the angle CEP, the triangles VHG, CEP will be similar; and thence it will come to pass that as EP is to CE so is HG to HV or HP, and so KI to KP, and by composition or division as CB to CE so is PI to PK, and doubling the consequents as CB to 2CE so PI to PV, and so is Pq to Pm. Therefore the decrement of the line VP, that is, the increment of the line BV — VP to the increment of the curve line AP is in a given ratio of CB to 2CE, and therefore (by cor. lem. 4) the lengths BV — VP and AP, generated by those increments, are in the same ratio. But if BV be radius, VP is the cosine of the angle BVP or ½BEI, and therefore BV — VP is the versed sine of the same angle, and therefore in this wheel, whose radius is ½BV, BV — VP will be double the versed sine of the arc ½BP. Therefore AP is to double the versed sine of the arc ½BP as 2CE to CB. Q.E.D.

The line AP in the former of these propositions we shall name the cycloid without the globe, the other in the latter proposition the cycloid within the globe, for distinction sake.

Cor. 1. Hence if there be described the entire cycloid ASL, and the same be bisected in S, the length of the part PS will be to the length PV (which is the double of the sine of the angle VBP, when EB is radius) as 2CE to CB, and therefore in a given ratio.

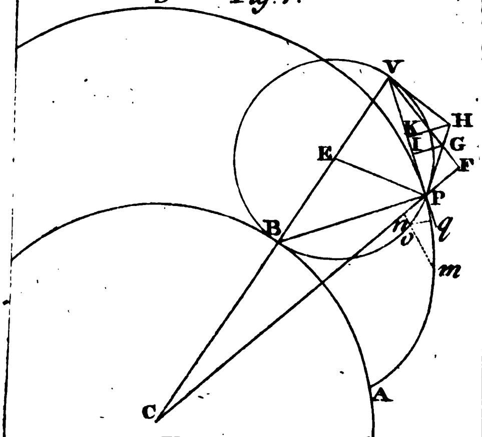
Cor. 2. And the length of the semi-perimeter of the cycloid AS will be equal to a right line which is to the diameter of the wheel BV as 2CE to CB.

PROPOSITION L. PROBLEM XXXIII.

To cause a pendulous body to oscillate in a given cycloid.

Let there be given within the globe QVS (Pl. 19, Fig. 3), described with the centre C, the cycloid QRS, bisected in R, and meeting the superficies of the globe with its extreme points Q and S on either hand. Let there be drawn CR bisecting the arc QS in O, and let it be produced to A in such sort that CA may be to CO as CO to CR. About the centre

Fig. 1.



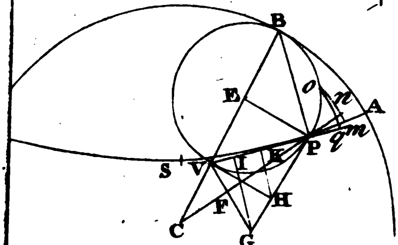
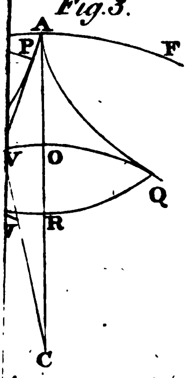


Fig. 3.



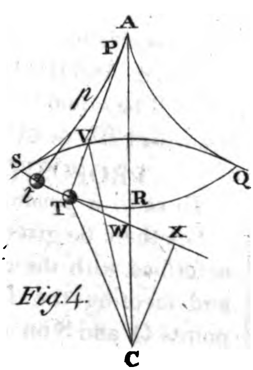


Fig. 4.

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C, with the interval CA, let there be described an exterior globe DAF; and within this globe, by a wheel whose diameter is AO, let there be described two semi-cycloids AQ, AS, touching the interior globe in Q and S, and meeting the exterior globe in A. From that point A, with a thread APT in length equal to the line AR, let the body T depend, and oscillate in such manner between the two semi-cyloids AQ, AS, that, as often as the pendulum parts from the perpendicular AR, the upper part of the thread AP may be applied to that semi-cycloid APS towards which the motion tends, and fold itself round that curve line, as if it were some solid obstacle, the remaining part of the same thread PT which has not yet touched the semi-cycloid continuing straight. Then will the weight T oscillate in the given cycloid QRS. Q.E.F.

For let the thread PT meet the cycloid QRS in T, and the circle QOS in V, and let CV be drawn; and to the rectilinear part of the thread PT from the extreme points P and T let there be erected the perpendiculars BP, TW, meeting the right line CV in B and W. It is evident, from the construction and generation of the similar figures AS, SR, that those perpendiculars PB, TW, cut off from CV the lengths VB, VW equal the diameters of the wheels OA, OR. Therefore TP is to VP (which is double the sine of the angle VBP when [2]BV is radius) as BW to BV, or AO + OR to AO, that is (since CA and CO, CO and CR, and by division AO and OR are proportional), as CA' + CO to CA, or, if BV be bisected in E, as 2CE to CB. Therefore (by cor. 1, prop. 49), the length of the rectilinear part of the thread PT is always equal to the arc of the cycloid PS, and the whole thread APT is always equal to the half of the cycloid APS, that is (by cor. 2, prop. 49), to the length AR. And therefore contrariwise, if the string remain always equal to the length AR, the point T will always move in the given cycloid QRS. Q.E.D.

Cor. The string AR is equal to the semi-cycloid AS, and therefore has the same ratio to AC the semi-diameter of the exterior globe as the like semi-cycloid SR has to CO the semi-diameter of the interior globe.

PROPOSITION LI. THEOREM XVIII.

If a centripetal force tending on all ſides to the centre C *of a globe* (Pl. 19, Fig. 4), *be in all places as the diſtance of the place from the centre, and by this force alone acting upon it, the body* T *oſcillate (in the manner above deſcribed) in the perimeter of the cycloid* QRS; *I ſay, that all the oſcillations, how unequal ſoever in themſelves, will be performed in equal times.*

For upon the tangent TW infinitely produced let fall the perpendicular CX, and join CT. Becauſe the centripetal force with which the body T is impelled towards C is as the diſtance CT, let this (by cor. 2, of the laws) be reſolved into the parts CX, TX, of which CX impelling the body directly from P ſtretches the thread PT, and by the reſiſtance the thread makes to it is totally employed, producing no other effect; but the other part TX, impelling the body tranſverſely or towards X, directly accelerates the motion in the cycloid. Then it is plain that the acceleration of the body, proportional to this accelerating force, will be every moment as the length TX, that is (becauſe CV, WV, and TX, TW proportional to them are given), as the length TW, that is (by cor. 1, prop. 49), as the length of the arc of the cycloid TR. If therefore two pendulums APT, Apt, be unequally drawn aſide from the perpendicular AR, and let fall together, their accelerations will be always as the arcs to be deſcribed TR, tR. But the parts deſcribed at the beginning of the motion are as the accelerations, that is, as the wholes that are to be deſcribed at the beginning, and therefore the parts which remain to be deſcribed, and the ſubſequent accelerations proportional to thoſe parts, are alſo as the wholes, and ſo on. Therefore the accelerations, and conſequently the velocities generated, and the parts deſcribed with thoſe velocities, and the parts to be deſcribed, are always as the wholes; and therefore the parts to be deſcribed preſerving a given ratio to each other will vaniſh together, that is, the two bodies oſcillating will arrive together at the perpendicular AR. And ſince on the other hand the aſcent of the pendulums from the loweſt place R through the ſame cycloidal arcs with a retrograde

motion, is retarded in the ſeveral places they paſs through by the ſame forces by which their deſcent was accelerated; it is plain that the velocities of their aſcent and deſcent through the ſame arcs are equal, and conſequently performed in equal times; and, therefore, ſince the two parts of the cycloid RS and RQ lying on either ſide of the perpendicular are ſimilar and equal, the two pendulums will perform as well the wholes as the halves of their oſcillations in the ſame times, Q.E.D.

Cor. The force with which the body T is accelerated or retarded in any place T of the cycloid, is to the whole weight of the ſame body in the higheſt place S or Q as the arc of the cycloid TR is to the arc SR or QR.

PROPOSITION LII. PROBLEM XXXIV.

To define the velocities of the pendulums in the ſeveral places, and the times in which both the entire oſcillations, and the ſeveral parts of them are performed.

About any centre G (Pl. 20, Fig. 1), with the interval GH equal to the arc of the cycloid RS, deſcribe a ſemi-circle HKM biſected by the ſemi-diameter GK. And if a centripetal force proportional to the diſtance of the places from the centre tend to the centre G, and it be in the perimeter HIK equal to the centripetal force in the perimeter of the globe QOS tending towards its centre, and at the ſame time that the pendulum T is let fall from the higheſt place S, a body, as L, is let fall from H to G; then becauſe the forces which act upon the bodies are equal at the beginning, and always proportional to the ſpaces to be deſcribed TR, LG, and therefore if TR and LG are equal, are alſo equal in the places T and L, it is plain that thoſe bodies deſcribe at the beginning equal ſpaces ST, HL, and therefore are ſtill acted upon equally, and continue to deſcribe equal ſpaces. Therefore by prop. 38, the time in which the body deſcribes the arc ST is to the time of one oſcillation, as the arc HI the time in which the body H arrives at L, to the ſemi-periphery HKM, the time in which the body H will come to M. And the velocity of the pendulous body in the place T is to its velocity in the loweſt place R, that is, the velocity of the

body H in the place L to its velocity in the place G, or the momentary increment of the line HL to the momentary increment of the line HG (the arcs HI, HK increafing with an equable flux) as the ordinate LI to the radius GK, or as $\sqrt{SR^2 - TR^2}$ to SR. Hence, fince in unequal ofcillations there are defcribed in equal time arcs proportional to the entire arcs of the ofcillations, there are obtained from the times given, both the velocities and the arcs defcribed in all the ofcillations univerfally. Which was firft required.

Let now any pendulous bodies ofcillate in different cycloids defcribed within different globes, whofe abfolute forces are alfo different; and if the abfolute force of any globe QOS be called V, the accelerative force with which the pendulum is acted on in the circumference of this globe, when it begins to move directly towards its centre, will be as the diftance of the pendulous body from that centre and the abfolute force of the globe conjunctly, that is, as $CO \times V$. Therefore the lineolæ HY, which is as this accelerated force $CO \times V$, will be defcribed in a given time; and if there be erected the perpendicular YZ meeting the circumference in Z, the nafcent arc HZ will denote that given time. But that nafcent arc HZ is in the fubduplicate ratio of the rectangle GHY, and therefore as $\sqrt{GH \times CO \times V}$. Whence the time of an entire ofcillation in the cycloid QRS (it being as the femi-periphery HKM, which denotes that entire ofcillation, directly; and as the arc HZ which in like manner denotes a given time inverfely) will be as GH directly and $\sqrt{GH \times CO \times V}$ inverfely; that is, becaufe GH and SR are equal, as $\sqrt{\frac{SR}{CO \times V}}$, or (by cor. prop. 50) as $\sqrt{\frac{AR}{AC \times V}}$. Therefore the ofcillations in all globes and cycloids, performed with what abfolute forces foever, are in a ratio compounded of the fubduplicate ratio of the length of the ftring directly, and the fubduplicate ratio of the diftance between the point of fufpenfion and the centre of the globe inverfely, and the fubdu-

plicate ratio of the abſolute force of the globe inverſely alſo. Q.E.I.

Cor. 1. Hence alſo the times of oſcillating, falling, and revolving bodies may be compared among themſelves. For if the diameter of the wheel with which the cycloid is deſcribed within the globe is ſuppoſed equal to the ſemi-diameter of the globe, the cycloid will become a right line paſſing through the centre of the globe, and the oſcillation will be changed into a deſcent and ſubſequent aſcent in that right line. Whence there is given both the time of the deſcent from any place to the centre, and the time equal to it in which the body revolving uniformly about the centre of the globe at any diſtance deſcribes an arc of a quadrant. For this time (by caſe 2) is to the time of half the oſcillation in any cycloid QRS as 1 to $\sqrt{\frac{AR}{AC}}$.

Cor. 2. Hence alſo follow what Sir *Chriſtopher Wren* and M. *Huygens* have diſcovered concerning the vulgar cycloid. For if the diameter of the globe be infinitely increaſed, its ſphærical ſuperficies will be changed into a plane, and the centripetal force will act uniformly in the direction of lines perpendicular to that plane, and this cycloid of our's will become the ſame with the common cycloid. But in that caſe the length of the arc of the cycloid between that plane and the deſcribing point will become equal to four times the verſed ſine of half the arc of the wheel between the ſame plane and the deſcribing point, as was diſcovered by Sir *Chriſtopher Wren*. And a pendulum between two ſuch cycloids will oſcillate in a ſimilar and equal cycloid in equal times, as M. *Huygens* demonſtrated. The deſcent of heavy bodies alſo in the time of one oſcillation will be the ſame as M. *Huygens* exhibited.

The propoſitions here demonſtrated are adapted to the true conſtitution of the Earth, in ſo far as wheels moving in any of its great circles will deſcribe, by the motions of nails fixed in their perimeters, cycloids without the globe; and pendulums in mines and deep caverns of the Earth, muſt oſcillate in cycloids within the globe, that thoſe oſcillations may be per-

formed in equal times. For gravity (as will be shewn in the third book) decreases in its progress from the superficies of the Earth; upwards in a duplicate ratio of the distances from the centre of the earth; downwards in a simple ratio of the same

PROPOSITION LIII. PROBLEM XXXV.

Granting the quadratures of curvilinear figures, it is required to find the forces with which bodies moving in given curve lines may always perform their oscillations in equal times.

Let the body T (Pl. 20, Fig. 2) oscillate in any curve line STRQ, whose axis is AR passing through the centre of force C. Draw TX touching that curve in any place of the body T, and in that tangent TX take TY equal to the arc TR. The length of that arc is known from the common methods used for the quadratures of figures. From the point Y draw the right line YZ perpendicular to the tangent. Draw CT meeting that perpendicular in Z, and the centripetal force will be proportional to the right line TZ. Q.E.I.

For if the force with which the body is attracted from T towards C be expressed by the right line TZ taken proportional to it, that force will be resolved into two forces TY, YZ, of which YZ drawing the body in the direction of the length of the thread PT, does not at all change its motion; whereas the other force TY directly accelerates or retards its motion in the curve STRQ. Wherefore since that force is as the space to be described TR, the accelerations or retardations of the body in describing two proportional parts (a greater and a less) of two oscillations, will be always as those parts, and therefore will cause those parts to be described together. But bodies which continually describe together parts proportional to the wholes, will describe the wholes together also. Q.E.D.

Cor. 1. Hence if the body T (Pl. 20, Fig. 3), hanging by a rectilinear thread AT from the centre A, describe the circular arc STRQ, and in the mean time be acted on by any force tending downwards with parallel directions, which is to the uniform force of gravity as the arc TR to its sine TN, the times of the several oscillations will be equal. For because

TZ, AR are parallel, the triangles ATN, ZTY are fimilar; and therefore TZ will be to AT as TY to TN; that is, if the uniform force of gravity be expreffed by the given length AT, the force TZ, by which the ofcillations become ifochronous, will be to the force of gravity AT, as the arc TR equal to TY is to TN the fine of that arc.

COR. 2. And therefore in clocks, if forces were impreffed by fome machine upon the pendulum which preferves the motion, and fo compounded with the force of gravity that the whole force tending downwards fhould be always as a line produced by applying the rectangle under the arc TR and the radius AR to the fine TN, all the ofcillations will become ifochronous.

PROPOSITION LIV. PROBLEM XXXVI.

Granting the quadratures of curvilinear figures, it is required to find the times in which bodies by means of any centripetal force will defcend or afcend in any curve lines defcribed in a plane paffing through the centre of force.

Let the body defcend from any place S (Pl. 20, Fig. 4), and move in any curve STtR given in a plane paffing through the centre of force C. Join CS, and let it be divided into innumerable equal parts, and let Dd be one of thofe parts. From the centre C, with the intervals CD, Cd, let the circles DT, dt be defcribed, meeting the curve line STtR in T and t. And becaufe the law of centripetal force is given, and alfo the altitude CS from which the body at firft fell, there will be given the velocity of the body in any other altitude CT (by prop. 39). But the time in which the body defcribes the lineolæ Tt is as the length of that lineolæ, that is, as the fecant of the angle tTC directly, and the velocity inverfely. Let the ordinate DN, proportional to this time, be made perpendicular to the right line CS at the point D, and becaufe Dd is given, the rectangle Dd × DN, that is, the area DNnd, will be proportional to the fame time. Therefore if PNn be a curve line in which the point N is perpetually found, and its afymptote be the right line SQ ftanding upon the line CS at right angles, the area SQPND will be proportional to the time in which the body in its defcent hath defcribed the

line ST; and therefore that area being found, the time is alfo given. Q.E.I.

PROPOSITION LV. THEOREM XIX.

If a body move in any curve fuperficies, whofe axis paffes through the centre of force, and from the body a perpendicular be let fall upon the axis; and a line parallel and equal thereto be drawn from any given point of the axis; I fay, that this parallel line will defcribe an area proportional to the time.

Let BKL (Pl. 20, Fig. 5) be a curve fuperficies, T a body revolving in it, STR a trajectory which the body defcribes in the fame, S the beginning of the trajectory, OMK the axis of the curve fuperficies, TN a right line let fall perpendicularly from the body to the axis; OP a line parallel and equal thereto drawn from the given point O in the axis; AP the orthographic projection of the trajectory defcribed by the point P in the plane AOP in which the revolving line OP is found; A the beginning of that projection, anfwering to the point S; TC a right line drawn from the body to the centre; TG a part thereof proportional to the centripetal force with which the body tends towards the centre C; TM a right line perpendicular to the curve fuperficies; TI a part thereof proportional to the force of preffure with which the body urges the fuperficies, and therefore with which it is again repelled by the fuperficies towards M; PTF a right line parallel to the axis and paffing through the body, and GF, IH right lines let fall perpendicularly from the points G and I upon that parallel PHTF. I fay, now, that the area AOP, defcribed by the radius OP from the beginning of the motion, is proportional to the time. For the force TG (by cor. 2, of the laws of motion) is refolved into the forces TF, FG; and the force TI into the forces TH, HI; but the forces TF, TH, acting in the direction of the line PF perpendicular to the plane AOP, introduce no change in the motion of the body but in a direction perpendicular to that plane. Therefore its motion, fo far as it has the fame direction with the pofition of the plane, that is, the motion of the point P, by which the projection AP of the trajectory is defcribed in that plane, is the fame as if the

forces TF, TH were taken away, and the body were acted on by the forces FG, HI alone; that is, the same as if the body were to describe in the plane AOP the curve AP by means of a centripetal force tending to the centre O, and equal to the sum of the forces FG and HI. But with such a force as that (by prop. 1) the area AOP will be described proportional to the time. Q.E.D.

COR. By the same reasoning, if a body, acted on by forces tending to two or more centres in any the same right line CO, should describe in a free space any curve line ST, the area AOP would be always proportional to the time.

PROPOSITION LVI. PROBLEM XXXVII.

Granting the quadratures of curvilinear figures, and supposing that there are given both the law of centripetal force tending to a given centre, and the curve superficies whose axis passes through that centre; it is required to find the trajectory which a body will describe in that superficies, when going off from a given place with a given velocity, and in a given direction in that superficies.

The last construction remaining, let the body T go from the given place S (Pl. 20, Fig. 6), in the direction of a line given by position, and turn into the trajectory sought STR, whose orthographic projection in the plane BLO is AP. And from the given velocity of the body in the altitude SC, its velocity in any other altitude TC will be also given. With that velocity, in a given moment of time, let the body describe the particle Tt of its trajectory, and let Pp be the projection of that particle described in the plane AOP. Join Op, and a little circle being described upon the curve superficies about the centre T with the interval Tt, let the projection of that little circle in the plane AOP be the ellipsis pQ. And because the magnitude of that little circle Tt, and TN or PO its distance from the axis CO is also given, the ellipsis pQ will be given both in kind and magnitude, as also its position to the right line PO. And since the area POp is proportional to the time, and therefore given because the time is given, the angle POp will be given. And thence will be given p the common intersection of the ellipsis and the right line Op, to-

gether with the angle OPp, in which the projection APp of the trajectory cuts the line OP. But from thence (by conferring prop. 41, with its 2d cor.) the manner of determining the curve APp easily appears. Then from the several points P of that projection erecting to the plane AOP, the perpendiculars PT meeting the curve superficies in T, there will be given the several points T of the trajectory. Q.E.I.

SECTION XI.

Of the motions of bodies tending to each other with centripetal forces.

I have hitherto been treating of the attractions of bodies towards an immovable centre; though very probably there is no such thing existent in nature. For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by law 3; so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both (by cor. 4, of the laws of motion), being as it were mutually attracted, revolve about a common centre of gravity. And if there be more bodies, which are either attracted by one single one which is attracted by them again, or which, all of them, attract each other mutually; these bodies will be so moved among themselves, as that their common centre of gravity will either be at rest, or move uniformly forward in a right line. I shall therefore at present go on to treat of the motion of bodies mutually attracting each other; considering the centripetal forces as attractions; though perhaps in a physical strictness they may more truly be called impulses. But these propositions are to be considered as purely mathematical; and therefore, laying aside all physical considerations, I make use of a familiar way of speaking, to make myself the more easily understood by a mathematical reader.

PROPOSITION LVII. THEOREM XX.

Two bodies attracting each other mutually describe similar figures about their common centre of gravity, and about each other mutually.

For the distances of the bodies from their common centre of gravity are reciprocally as the bodies; and therefore in a

given ratio to each other; and thence, by composition of ratios, in a given ratio to the whole distance between the bodies. Now these distances revolve about their common term with an equable angular motion, because lying in the same right line they never change their inclination to each other mutually. But right lines that are in a given ratio to each other, and revolve about their terms with an equal angular motion, describe upon planes, which either rest with those terms, or move with any motion not angular, figures entirely similar round those terms. Therefore the figures described by the revolution of these distances are similar. Q.E.D.

PROPOSITION LVIII. THEOREM XXI.

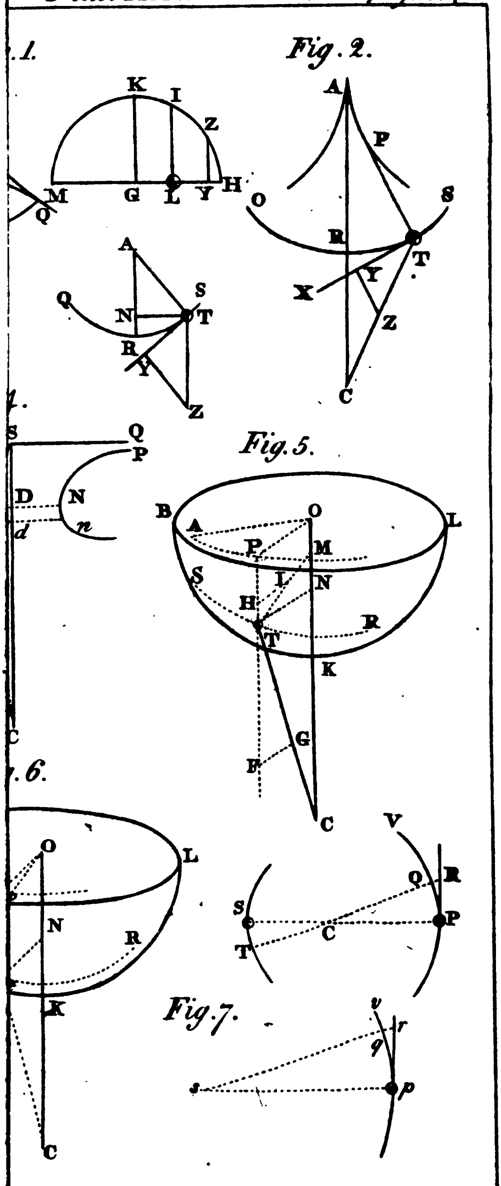
If two bodies attract each other mutually with forces of any kind, and in the mean time revolve about the common centre of gravity; I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other mutually.

Let the bodies S and P (Pl. 20, Fig. 7) revolve about their common centre of gravity C, proceeding from S to T, and from P to Q. From the given point s let there be continually drawn sp, sq, equal and parallel to SP, TQ; and the curve pqv, which the point p describes in its revolution round the immovable point s, will be similar and equal to the curves which the bodies S and P describe about each other mutually; and therefore, by theor. 20, similar to the curves ST and PQV which the same bodies describe about their common centre of gravity C; and that because the proportions of the lines SC, CP, and SP or sp, to each other, are given.

Case 1. The common centre of gravity C (by cor. 4, of the laws of motion) is either at rest, or moves uniformly in a right line. Let us first suppose it at rest, and in s and p let there be placed two bodies, one immovable in s, the other moveable in p, similar and equal to the bodies S and P. Then let the right lines PR and pr touch the curves PQ and pq in P and p, and produce CQ and sq to R and r. And because the figures CPRQ, sprq are similar, RQ will be to rq as CP to sp, and therefore in a given ratio. Hence if the force with

which the body P is attracted towards the body S, and by consequence towards the intermediate point the centre C, were to the force with which the body p is attracted towards the centre s, in the same given ratio, these forces would in equal times attract the bodies from the tangents PR, pr to the arcs PQ, pq, through the intervals proportional to them RQ, rq; and therefore this last force (tending to s) would make the body p revolve in the curve pqv, which would become similar to the curve PQV, in which the first force obliges the body P to revolve; and their revolutions would be completed in the same times. But because those forces are not to each other in the ratio of CP to sp, but (by reason of the similarity and equality of the bodies S and s, P and p, and the equality of the distances SP, sp) mutually equal, the bodies in equal times will be equally drawn from the tangents; and therefore that the body p may be attracted through the greater interval rq, there is required a greater time, which will be in the subduplicate ratio of the intervals; because, by lemma 10, the spaces described at the very beginning of the motion are in a duplicate ratio of the times. Suppose, then, the velocity of the body p to be to the velocity of the body P in a subduplicate ratio of the distance sp to the distance CP, so that the arcs pq, PQ, which are in a simple proportion to each other, may be described in times that are in a subduplicate ratio of the distances; and the bodies P, p, always attracted by equal forces, will describe round the quiescent centres C and s similar figures PQV, pqv, the latter of which pqv is similar and equal to the figure which the body P describes round the moveable body S. Q.E.D.

CASE 2. Suppose now that the common centre of gravity, together with the space in which the bodies are moved among themselves, proceeds uniformly in a right line; and (by cor. 6, of the laws of motion) all the motions in this space will be performed in the same manner as before; and therefore the bodies will describe mutually about each other the same figures as before, which will be therefore similar and equal to the figure pqv. Q.E.D.


.1.
Fig. 2.
4.
Fig. 5.
.6.
Fig. 7.

Cor. 1. Hence two bodies attracting each other with forces proportional to their diſtance, deſcribe (by prop. 10) both round their common centre of gravity, and round each other mutually, concentrical ellipſes; and, *vice verſa,* if ſuch figures are deſcribed, the forces are proportional to the diſtances.

Cor. 2. And two bodies, whoſe forces are reciprocally proportional to the ſquare of their diſtance, deſcribe (by prop. 11, 12, 13), both round their common centre of gravity, and round each other mutually, conic ſections having their focus in the centre about which the figures are deſcribed. And, *vice verſa,* if ſuch figures are deſcribed, the centripetal forces are reciprocally proportional to the ſquares of the diſtance.

Cor. 3. Any two bodies revolving round their common centre of gravity deſcribe areas proportional to the times, by radii drawn both to that centre and to each other mutually.

PROPOSITION LIX. THEOREM XXII.

The periodic time of two bodies S *and* P *revolving round their common centre of gravity* C, *is to the periodic time of one of the bodies* P *revolving round the other* S *remaining unmoved, and deſcribing a figure ſimilar and equal to thoſe which the bodies deſcribe about each other mutually, in a ſubduplicate ratio of the other body* S *to the ſum of the bodies* S + P.

For, by the demonſtration of the laſt propoſition, the times in which any ſimilar arcs PQ and pq are deſcribed are in a ſubduplicate ratio of the diſtances CP and SP, or sp, that is, in a ſubduplicate ratio of the body S to the ſum of the bodies S + P. And by compoſition of ratios, the ſums of the times in which all the ſimilar arcs PQ and pq are deſcribed, that is, the whole times in which the whole ſimilar figures are deſcribed, are in the ſame ſubduplicate ratio. Q.E.D.

PROPOSITION LX. THEOREM XXIII.

If two bodies S *and* P, *attracting each other with forces reciprocally proportional to the ſquares of their diſtance, revolve about their common centre of gravity; I ſay, that the principal axis of the ellipſis which either of the bodies, as* P, *deſcribes by this motion about the other* S, *will be to the principal axis of the ellipſis, which the ſame body* P *may deſcribe*

in the same periodical time about the other body S *quiescent, as the sum of the two bodies* S + P *to the first of two mean proportionals between that sum and the other body* S.

For if the ellipses described were equal to each other, their periodic times by the last theorem would be in a subduplicate ratio of the body S to the sum of the bodies S + P. Let the periodic time in the latter ellipsis be diminished in that ratio, and the periodic times will become equal; but, by prop. 15, the principal axis of the ellipsis will be diminished in a ratio sesquiplicate to the former ratio; that is, in a ratio to which the ratio of S to S + P is triplicate; and therefore that axis will be to the principal axis of the other ellipsis as the first of two mean proportionals between S + P and S to S + P. And inversely the principal axis of the ellipsis described about the moveable body will be to the principal axis of that described round the immovable as S + P to the first of two mean proportionals between S + P and S. Q.E.D.

PROPOSITION LXI. THEOREM XXIV.

If two bodies attracting each other with any kind of forces, and not otherwise agitated or obstructed, are moved in any manner whatsoever, those motions will be the same as if they did not at all attract each other mutually, but were both attracted with the same forces by a third body placed in their common centre of gravity; and the law of the attracting forces will be the same in respect of the distance of the bodies from the common centre, as in respect of the distance between the two bodies.

For those forces with which the bodies attract each other mutually, by tending to the bodies, tend also to the common centre of gravity lying directly between them; and therefore are the same as if they proceeded from an intermediate body. Q.E.D.

And because there is given the ratio of the distance of either body from that common centre to the distance between the two bodies, there is given, of course, the ratio of any power of one distance to the same power of the other distance; and also the ratio of any quantity derived in any manner from one of the distances compounded any how with given quantities, to another quantity derived in like manner from the

other distance, and as many given quantities having that given ratio of the distances to the first. Therefore if the force with which one body is attracted by another be directly or inversely as the distance of the bodies from each other, or as any power of that distance; or, lastly, as any quantity derived after any manner from that distance compounded with given quantities; then will the same force with which the same body is attracted to the common centre of gravity be in like manner directly or inversely as the distance of the attracted body from the common centre, or as any power of that distance; or, lastly, as a quantity derived in like sort from that distance compounded with analogous given quantities. That is, the law of attracting force will be the same with respect to both distances, Q.E.D.

PROPOSITION LXII. PROBLEM XXXVIII.

To determine the motions of two bodies which attract each other with forces reciprocally proportional to the squares of the distance between them, and are let fall from given places.

The bodies, by the last theorem, will be moved in the same manner as if they were attracted by a third placed in the common centre of their gravity; and by the hypothesis that centre will be quiescent at the beginning of their motion, and therefore (by cor. 4, of the laws of motion) will be always quiescent. The motions of the bodies are therefore to be determined (by prob. 25) in the same manner as if they were impelled by forces tending to that centre; and then we shall have the motions of the bodies attracting each other mutually. Q.E.I.

PROPOSITION LXIII. PROBLEM XXXIX.

To determine the motions of two bodies attracting each other with forces reciprocally proportional to the squares of their distance, and going off from given places in given directions with given velocities.

The motions of the bodies at the beginning being given, there is given also the uniform motion of the common centre of gravity, and the motion of the space which moves along with this centre uniformly in a right line, and also the very first, or beginning motions of the bodies in respect of this

ſpace. Then (by cor. 5, of the laws, and the laſt theorem) the ſubſequent motions will be performed in the ſame manner in that ſpace, as if that ſpace together with the common centre of gravity were at reſt, and as if the bodies did not attract each other, but were attracted by a third body placed in that centre. The motion therefore in this moveable ſpace of each body going off from a given place, in a given direction, with a given velocity, and acted upon by a centripetal force tending to that centre, is to be determined by prob. 9 and 26, and at the ſame time will be obtained the motion of the other round the ſame centre. With this motion compound the uniform progreſſive motion of the entire ſyſtem of the ſpace and the bodies revolving in it, and there will be obtained the abſolute motion of the bodies in immovable ſpace. Q.E.I.

PROPOSITION LXIV. PROBLEM XL.

Suppoſing forces with which bodies mutually attract each other to increaſe in a ſimple ratio of their diſtances from the centres; it is required to find the motions of ſeveral bodies among themſelves.

Suppoſe the two firſt bodies T and L (Pl. 21, Fig. 1) to have their common centre of gravity in D. Theſe, by cor. 1, theor. 21, will deſcribe ellipſes having their centres in D, the magnitudes of which ellipſes are known by prob. 5.

Let now a third body S attract the two former T and L with the accelerative forces ST, SL, and let it be attracted again by them. The force ST (by cor. 2, of the laws of motion) is reſolved into the forces SD, DT; and the force SL into the forces SD and DL. Now the forces DT, DL, which are as their ſum TL, and therefore as the accelerative forces with which the bodies T and L attract each other mutually, added to the forces of the bodies T and L, the firſt to the firſt, and the laſt to the laſt, compoſe forces proportional to the diſtances DT and DL as before, but only greater than thoſe former forces; and therefore (by cor 1, prop. 10, and cor. 1, and 8, prop. 4) they will cauſe thoſe bodies to deſcribe ellipſes as before, but with a ſwifter motion. The remaining accelerative forces SD and DL, by the motive forces SD × T and SD × L, which are as the bodies attracting thoſe bodies

equally, and in the direction of the lines TI, LK parallel to DS, do not at all change their situations with respect to one another, but cause them equally to approach to the line IK; which must be imagined drawn through the middle of the body S, and perpendicular to the line DS. But that approach to the line IK will be hindered by causing the system of the bodies T and L on one side, and the body S on the other, with proper velocities, to revolve round the common centre of gravity C. With such a motion the body S, because the sum of the motive forces SD × T and SD × L is proportional to the distance CS, tends to the centre C, will describe an ellipsis round the same centre C; and the point D, because the lines CS and CD are proportional, will describe a like ellipsis over against it. But the bodies T and L, attracted by the motive forces SD × T and SD × L, the first by the first, and the last by the last, equally and in the direction of the parallel lines TI and LK, as was said before, will (by cor. 5 and 6, of the laws of motion) continue to describe their ellipses round the moveable centre D, as before. Q.E.I.

Let there be added a fourth body V, and, by the like reasoning, it will be demonstrated that this body and the point C will describe ellipses about the common centre of gravity B; the motions of the bodies T, L, and S round the centres D and C remaining the same as before; but accelerated. And by the same method one may add yet more bodies at pleasure. Q.E.I.

This would be the case, though the bodies T and L attract each other mutually with accelerative forces either greater or less than those with which they attract the other bodies in proportion to their distance. Let all the mutual accelerative attractions be to each other as the distances multiplied into the attracting bodies; and from what has gone before it will easily be concluded that all the bodies will describe different ellipses with equal periodical times about their common centre of gravity B, in an immovable plane. Q.E.I.

PROPOSITION LXV. THEOREM XXV.

Bodies, whose forces decrease in a duplicate ratio of their distances from their centres, may move among themselves in

ellipses; and by radii drawn to the foci may describe areas proportional to the times very nearly.

In the last proposition we demonstrated that case in which the motions will be performed exactly in ellipses. The more distant the law of the forces is from the law in that case, the more will the bodies disturb each other's motions; neither is it possible that bodies attracting each other mutually according to the law supposed in this proposition should move exactly in ellipses, unless by keeping a certain proportion of distances from each other. However, in the following cases the orbits will not much differ from ellipses.

CASE 1. Imagine several lesser bodies to revolve about some very great one at different distances from it, and suppose absolute forces tending to every one of the bodies proportional to each. And because (by cor. 4, of the laws) the common centre of gravity of them all is either at rest, or moves uniformly forward in a right line, suppose the lesser bodies so small that the great body may be never at a sensible distance from that centre; and then the great body will, without any sensible error, be either at rest, or move uniformly forward in a right line; and the lesser will revolve about that great one in ellipses, and by radii drawn thereto will describe areas proportional to the times; if we except the errors that may be introduced by the receding of the great body from the common centre of gravity, or by the mutual actions of the lesser bodies upon each other. But the lesser bodies may be so far diminished, as that this recess and the mutual actions of the bodies on each other may become less than any assignable; and therefore so as that the orbits may become ellipses, and the areas answer to the times, without any error that is not less than any assignable. Q.E.O.

CASE 2. Let us imagine a system of lesser bodies revolving about a very great one in the manner just described, or any other system of two bodies revolving about each other to be moving uniformly forward in a right line, and in the mean time to be impelled sideways by the force of another vastly greater body situate at a great distance. And because the equal accelerative forces with which the bodies are impelled

in parallel directions do not change the situation of the bodies with respect to each other, but only oblige the whole system to change its place while the parts still retain their motions among themselves, it is manifest that no change in those motions of the attracted bodies can arise from their attractions towards the greater, unless by the inequality of the accelerative attractions, or by the inclinations of the lines towards each other, in whose directions the attractions are made. Suppose, therefore, all the accelerative attractions made towards the great body to be among themselves as the squares of the distances reciprocally; and then, by increasing the distance of the great body 'till the differences of the right lines drawn from that to the others in respect of their length, and the inclinations of those lines to each other, be less than any given, the motions of the parts of the system will continue without errors that are not less than any given. And because, by the small distance of those parts from each other, the whole system is attracted as if it were but one body, it will therefore be moved by this attraction as if it were one body; that is, its centre of gravity will describe about the great body one of the conic sections (that is, a parabola or hyperbola when the attraction is but languid, and an ellipsis when it is more vigorous); and by radii drawn thereto, it will describe areas proportional to the times, without any errors but those which arise from the distances of the parts, which are by the supposition exceedingly small, and may be diminished at pleasure. Q.E.O.

By a like reasoning one may proceed to more compounded cases *in infinitum.*

Cor. 1. In the second case, the nearer the very great body approaches to the system of two or more revolving bodies, the greater will the perturbation be of the motions of the parts of the system among themselves; because the inclinations of the lines drawn from that great body to those parts become greater; and the inequality of the proportion is also greater.

Cor. 2. But the perturbation will be greatest of all, if we suppose the accelerative attractions of the parts of the system towards the greatest body of all are not to each other reci-

procally as the squares of the distances from that great body; especially if the inequality of this proportion be greater than the inequality of the proportion of the distances from the great body. For if the accelerative force, acting in parallel directions and equally, causes no perturbation in the motions of the parts of the system, it must of course, when it acts unequally, cause a perturbation somewhere, which will be greater or less as the inequality is greater or less. The excess of the greater impulses acting upon some bodies, and not acting upon others, must necessarily change their situation among themselves. And this perturbation, added to the perturbation arising from the inequality and inclination of the lines, makes the whole perturbation greater.

COR. 3. Hence if the parts of this system move in ellipses or circles without any remarkable perturbation, it is manifest that, if they are at all impelled by accelerative forces tending to any other bodies, the impulse is very weak, or else is impressed very near equally and in parallel directions upon all of them.

PROPOSITION LXVI. THEOREM XXVI.

If three bodies whose forces decrease in a duplicate ratio of the distances attract each other mutually; and the accelerative attractions of any two towards the third be between themselves reciprocally as the squares of the distances; and the two least revolve about the greatest; I say, that the interior of the two revolving bodies will, by radii drawn to the innermost and greatest, describe round that body areas more proportional to the times, and a figure more approaching to that of an ellipsis having its focus in the point of concourse of the radii, if that great body be agitated by those attractions, than it would do if that great body were not attracted at all by the lesser, but remained at rest; or than it would if that great body were very much more or very much less attracted, or very much more or very much less agitated, by the attractions.

This appears plainly enough from the demonstration of the second corollary of the foregoing proposition; but it may be

made out after this manner by a way of reafoning more diftinct and more univerfally convincing.

CASE 1. Let the leffer bodies P and S (Pl. 21, Fig. 2) revolve in the fame plane about the greateft body T, the body P defcribing the interior orbit PAB, and S the exterior orbit ESE. Let SK be the mean diftance of the bodies P and S; and let the accelerative attraction of the body P towards S, at that mean diftance, be expreffed by that line SK. Make SL to SK as the fquare of SK to the fquare of SP, and SL will be the accelerative attraction of the body P towards S at any diftance SP. Join PT, and draw LM parallel to it meeting ST in M; and the attraction SL will be refolved (by cor. 2, of the laws of motion) into the attractions SM, LM. And fo the body P will be urged with a threefold accelerative force. One of thefe forces tends towards T, and arifes from the mutual attraction of the bodies T and P. By this force alone the body P would defcribe round the body T, by the radius PT, areas proportional to the times, and an ellipfis whofe focus is in the centre of the body T; and this it would do whether the body T remained unmoved, or whether it were agitated by that attraction. This appears from prop. 11, and cor. 2 and 3 of theor. 21. The other force is that of the attraction LM, which, becaufe it tends from P to T, will be fuperadded to and coincide with the former force; and caufe the areas to be ftill proportional to the times, by cor. 3, theor. 21. But becaufe it is not reciprocally proportional to the fquare of the diftance PT, it will compofe, when added to the former, a force varying from that proportion; which variation will be the greater by how much the proportion of this force to the former is greater, *cæteris paribus.* Therefore, fince by prop. 11, and by cor. 2, theor. 21, the force with which the ellipfis is defcribed about the focus T ought to be directed to that focus, and to be reciprocally proportional to the fquare of the diftance PT, that compounded force varying from that proportion will make the orbit PAB vary from the figure of an ellipfis that has its focus in the point T; and fo much the more by how much the variation from that proportion is greater; and by confequence by how much the

proportion of the second force LM to the first force is greater, *cæteris paribus*. But now the third force SM, attracting the body P in a direction parallel to ST, composes with the other forces a new force which is no longer directed from P to T; and which varies so much more from this direction by how much the proportion of this third force to the other forces is greater, *cæteris paribus*; and therefore causes the body P to describe, by the radius TP, areas no longer proportional to the times; and therefore makes the variation from that proportionality so much greater by how much the proportion of this force to the others is greater. But this third force will increase the variation of the orbit PAB from the elliptical figure before-mentioned upon two accounts; first, because that force is not directed from P to T; and, secondly, because it is not reciprocally proportional to the square of the distance *PT*. These things being premised, it is manifest that the areas are then most nearly proportional to the times, when that third force is the least possible, the rest preserving their former quantity; and that the orbit PAB does then approach nearest to the elliptical figure above-mentioned, when both the second and third, but especially the third force, is the least possible; the first force remaining in its former quantity.

Let the accelerative attraction of the body T towards S be expressed by the line SN; then if the accelerative attractions SM and SN were equal, these, attracting the bodies T and P equally and in parallel directions, would not at all change their situation with respect to each other. The motions of the bodies between themselves would be the same in that case as if those attractions did not act at all, by cor. 6, of the laws of motion. And, by a like reasoning, if the attraction SN is less than the attraction SM, it will take away out of the attraction SM the part SN, so that there will remain only the part (of the attraction) MN to disturb the proportionality of the areas and times, and the elliptical figure of the orbit. And in like manner if the attraction SN be greater than the attraction SM, the perturbation of the orbit and proportion will be produced by the difference MN alone. After this manner the attraction SN reduces always the attraction SM to the at-

traction MN, the first and second attractions remaining perfectly unchanged; and therefore the areas and times come then nearest to proportionality, and the orbit PAB to the above-mentioned elliptical figure, when the attraction MN is either none, or the least that is possible; that is, when the accelerative attractions of the bodies P and T approach as near as possible to equality; that is, when the attraction SN is neither none at all, nor less than the least of all the attractions SM, but is, as it were, a mean between the greatest and least of all those attractions SM, that is, not much greater nor much less than the attraction SK. Q.E.D.

Case 2. Let now the lesser bodies P, S, revolve about a greater T in different planes; and the force LM, acting in the direction of the line PT situate in the plane of the orbit PAB, will have the same effect as before; neither will it draw the body P from the plane of its orbit. But the other force NM acting in the direction of a line parallel to ST (and which, therefore, when the body S is without the line of the nodes is inclined to the plane of the orbit PAB), besides the perturbation of the motion just now spoken of as to longitude, introduces another perturbation also as to latitude, attracting the body P out of the plane of its orbit. And this perturbation, in any given situation of the bodies P and T to each other, will be as the generating force MN; and therefore becomes least when the force MN is least, that is (as was just now shewn), where the attraction SN is not much greater nor much less than the attraction SK. Q.E.D.

Cor. 1. Hence it may be easily collected, that if several less bodies P, S, R, &c. revolve about a very great body T, the motion of the innermost revolving body P will be least disturbed by the attractions of the others, when the great body is as well attracted and agitated by the rest (according to the ratio of the accelerative forces) as the rest are by each other mutually.

Cor. 2. In a system of three bodies T, P, S, if the accelerative attractions of any two of them towards a third be to each other reciprocally as the squares of the distances, the body P, by the radius PT, will describe its area swifter near

the conjunction A and the opposition B than it will near the quadratures C and D. For every force with which the body P is acted on and the body T is not, and which does not act in the direction of the line PT, does either accelerate or retard the description of the area, according as it is directed, whether *in consequentia* or *in antecedentia*. Such is the force NM. This force in the passage of the body P from C to A is directed *in consequentia* to its motion, and therefore accelerates it; then as far as D *in antecedentia*, and retards the motion; then *in consequentia* as far as B; and lastly *in antecedentia* as it moves from B to C.

Cor. 3. And from the same reasoning it appears that the body P, *cæteris paribus*, moves more swiftly in the conjunction and opposition than in the quadratures.

Cor. 4. The orbit of the body P, *cæteris paribus*, is more curve at the quadratures than at the conjunction and opposition. For the swifter bodies move, the less they deflect from a rectilinear path. And besides the force KL, or NM, at the conjunction and opposition, is contrary to the force with which the body T attracts the body P, and therefore diminishes that force; but the body P will deflect the less from a rectilinear path the less it is impelled towards the body T.

Cor. 5. Hence the body P, *cæteris paribus*, goes farther from the body T at the quadratures than at the conjunction and opposition. This is said, however, supposing no regard had to the motion of eccentricity. For if the orbit of the body P be eccentrical, its eccentricity (as will be shewn presently by cor. 9) will be greatest when the apsides are in the syzygies; and thence it may sometimes come to pass that the body P, in its near approach to the farther apsis, may go farther from the body T at the syzygies than at the quadratures.

Cor. 6. Because the centripetal force of the central body T, by which the body P is retained in its orbit, is increased at the quadratures by the addition caused by the force LM, and diminished at the syzygies by the subduction caused by the force KL; and by reason the force KL is greater than

LM is more diminiſhed than increaſed; and, moreover, ſince that centripetal force (by cor. 2, prop. 4) is in a ratio compounded of the ſimple ratio of the radius TP directly, and the duplicate ratio of the periodical time inverſely; it is plain that this compounded ratio is diminiſhed by the action of the force KL; and therefore that the periodical time, ſuppoſing the radius of the orbit PT to remain the ſame, will be increaſed, and that in the ſubduplicate of that ratio in which the centripetal force is diminiſhed; and therefore, ſuppoſing this radius increaſed or diminiſhed, the periodical time will be increaſed more or diminiſhed leſs than in the ſeſquiplicate ratio of this radius, by cor. 6, prop. 4. If that force of the central body ſhould gradually decay, the body P being leſs and leſs attracted would go farther and farther from the centre T; and, on the contrary, if it were increaſed, it would draw nearer to it. Therefore if the action of the diſtant body S, by which that force is diminiſhed, were to increaſe and decreaſe by turns, the radius TP will be alſo increaſed and diminiſhed by turns; and the periodical time will be increaſed and diminiſhed in a ratio compounded of the ſeſquiplicate ratio of the radius, and of the ſubduplicate of that ratio in which the centripetal force of the central body T is diminiſhed or increaſed, by the increaſe or decreaſe of the action of the diſtant body S.

Cor. 7. It alſo follows, from what was before laid down, that the axis of the ellipſis deſcribed by the body P, or the line of the apſides, does as to its angular motion go forwards and backwards by turns, but more forwards than backwards, and by the exceſs of its direct motion is in the whole carried forwards. For the force with which the body P is urged to the body T at the quadratures, where the force MN vaniſhes, is compounded of the force LM and the centripetal force with which the body T attracts the body P. The firſt force LM, if the diſtance PT be increaſed, is increaſed in nearly the ſame proportion with that diſtance, and the other force decreaſes in the duplicate ratio of the diſtance; and therefore the ſum of theſe two forces decreaſes in a leſs than the dupli-

cate ratio of the diſtance PT; and therefore, by cor. 1, prop. 45, will make the line of the apſides, or, which is the ſame thing, the upper apſis, to go backward. But at the conjunction and oppoſition the force with which the body P is urged towards the body T is the difference of the force KL, and of the force with which the body T attracts the body P; and that difference, becauſe the force KL is very nearly increaſed in the ratio of the diſtance PT, decreaſes in more than the duplicate ratio of the diſtance PT; and therefore, by cor. 1, prop. 45, cauſes the line of the apſides to go forwards. In the places between the ſyzygies and the quadratures, the motion of the line of the apſides depends upon both theſe cauſes conjunctly, ſo that it either goes forwards or backwards in proportion to the exceſs of one of theſe cauſes above the other. Therefore ſince the force KL in the ſyzygies is almoſt twice as great as the force LM in the quadratures, the exceſs will be on the ſide of the force KL, and by conſequence the line of the apſides will be carried forwards. The truth of this and the foregoing corollary will be more eaſily underſtood by conceiving the ſyſtem of the two bodies T and P to be ſurrounded on every ſide by ſeveral bodies S, S, S, &c. diſpoſed about the orbit ESE. For by the actions of theſe bodies the action of the body T will be diminiſhed on every ſide, and decreaſe in more than a duplicate ratio of the diſtance.

COR. 8. But ſince the progreſs or regreſs of the apſides depends upon the decreaſe of the centripetal force, that is, upon its being in a greater or leſs ratio than the duplicate ratio of the diſtance TP, in the paſſage of the body from the lower apſis to the upper; and upon a like increaſe in its return to the lower apſis again; and therefore becomes greateſt where the proportion of the force at the upper apſis to the force at the lower apſis recedes fartheſt from the duplicate ratio of the diſtances inverſely; it is plain, that, when the apſides are in the ſyzygies, they will, by reaſon of the ſubducting force KL or NM — LM, go forward more ſwiftly; and in the quadratures by the additional force LM go backward more ſlowly. Becauſe the velocity of the progreſs or ſlow-

nefs of the regrefs is continued for a long time; this inequality becomes exceedingly great.

Cor. 9. If a body is obliged, by a force reciprocally proportional to the fquare of its diftance from any centre, to revolve in an ellipfis round that centre; and afterwards in its defcent from the upper apfis to the lower apfis, that force by a perpetual acceffion of new force is increafed in more than a duplicate ratio of the diminifhed diftance; it is manifeft that the body, being impelled always towards the centre by the perpetual acceffion of this new force, will incline more towards that centre than if it were urged by that force alone which decreafes in a duplicate ratio of the diminifhed diftance, and therefore will defcribe an orbit interior to that elliptical orbit, and at the lower apfis approaching nearer to the centre than before. Therefore the orbit by the acceffion of this new force will become more eccentrical. If now, while the body is returning from the lower to the upper apfis, it fhould decreafe by the fame degrees by which it increafes before the body would return to its firft diftance; and therefore if the force decreafes in a yet greater ratio, the body, being now lefs attracted than before, will afcend to a ftill greater diftance, and fo the eccentricity of the orbit will be increafed ftill more. Therefore if the ratio of the increafe and decreafe of the centripetal force be augmented each revolution, the eccentricity will be augmented alfo; and, on the contrary, if that ratio decreafe, it will be diminifhed.

Now, therefore, in the fyftem of the bodies T, P, S, when the apfides of the orbit PAB are in the quadratures, the ratio of that increafe and decreafe is leaft of all, and becomes greateft when the apfides are in the fyzygies. If the apfides are placed in the quadratures, the ratio near the apfides is lefs, and near the fyzygies greater, than the duplicate ratio of the diftances; and from that greater ratio arifes a direct motion of the line of the apfides, as was juft now faid. But if we confider the ratio of the whole increafe or decreafe in the progrefs between the apfides, this is lefs than the duplicate ratio of the diftances. The force in the lower is to the force in the upper apfis in lefs than a duplicate ratio of the diftance of

the upper apsis from the focus of the ellipsis to the distance of the lower apsis from the same focus; and, contrariwise, when the apsides are placed in the syzygies, the force in the lower apsis is to the force in the upper apsis in a greater than a duplicate ratio of the distances. For the forces LM in the quadratures added to the forces of the body T compose forces in a less ratio; and the forces KL in the syzygies subducted from the forces of the body T, leave the forces in a greater ratio. Therefore the ratio of the whole increase and decrease in the passage between the apsides is least at the quadratures and greatest at the syzygies; and therefore in the passage of the apsides from the quadratures to the syzygies it is continually augmented, and increases the eccentricity of the ellipsis; and in the passage from the syzygies to the quadratures it is perpetually decreasing, and diminishes the eccentricity.

Cor. 10. That we may give an account of the errors as to latitude, let us suppose the plane of the orbit EST to remain immovable; and from the cause of the errors above explained, it is manifest, that, of the two forces NM, ML, which are the only and entire cause of them, the force ML acting always in the plane of the orbit PAB never disturbs the motions as to latitude; and that the force NM, when the nodes are in the syzygies, acting also in the same plane of the orbit, does not at that time affect those motions. But when the nodes are in the quadratures, it disturbs them very much, and, attracting the body P perpetually out of the plane of its orbit, it diminishes the inclination of the plane in the passage of the body from the quadratures to the syzygies, and again increases the same in the passage from the syzygies to the quadratures. Hence it comes to pass that when the body is in the syzygies, the inclination is then least of all, and returns to the first magnitude nearly, when the body arrives at the next node. But if the nodes are situate at the octants after the quadratures, that is, between C and A, D and B, it will appear, from what was just now shewn, that, in the passage of the body P from either node to the ninetieth degree from thence, the inclination of the plane is perpetually diminished; then, in the passage

through the next 45 degrees to the next quadrature, the inclination is increased; and afterwards, again, in its passage through another 45 degrees to the next node, it is diminished. Therefore the inclination is more diminished than increased, and is therefore always less in the subsequent node than in the preceding one. And, by a like reasoning, the inclination is more increased than diminished when the nodes are in the other octants between A and D, B and C. The inclination, therefore, is the greatest of all when the nodes are in the syzygies. In their passage from the syzygies to the quadratures the inclination is diminished at each appulse of the body to the nodes; and becomes least of all when the nodes are in the quadratures, and the body in the syzygies; then it increases by the same degrees by which it decreased before; and, when the nodes come to the next syzygies, returns to its former magnitude.

Cor. 11. Because when the nodes are in the quadratures the body P is perpetually attracted from the plane of its orbit; and because this attraction is made towards S in its passage from the node C through the conjunction A to the node D; and to the contrary part in its passage from the node D through the opposition B to the node C; it is manifest that, in its motion from the node C, the body recedes continually from the former plane CD of its orbit till it comes to the next node; and therefore at that node, being now at its greatest distance from the first plane CD, it will pass through the plane of the orbit EST not in D, the other node of that plane, but in a point that lies nearer to the body S, which therefore becomes a new place of the node *in antecedentia* to its former place. And, by a like reasoning, the nodes will continue to recede in their passage from this node to the next. The nodes, therefore, when situate in the quadratures, recede perpetually; and at the syzygies, where no perturbation can be produced in the motion as to latitude, are quiescent: in the intermediate places they partake of both conditions, and recede more slowly; and, therefore, being always either retrograde or stationary, they will be carried backwards, or *in antecedentia*, each revolution.

COR. 12. All the errors defcribed in thefe corollaries are a little greater at the conjunction of the bodies P, S, than at their oppofition; becaufe the generating forces NM and ML are greater.

COR. 13. And fince the caufes and proportions of the errors and variations mentioned in thefe corollaries do not depend upon the magnitude of the body S, it follows that all things before demonftrated will happen, if the magnitude of the body S be imagined fo great as that the fyftem of the two bodies P and T may revolve about it. And from this increafe of the body S, and the confequent increafe of its centripetal force, from which the errors of the body P arife, it will follow that all thefe errors, at equal diftances, will be greater in this cafe, than in the other where the body S revolves about the fyftem of the bodies P and T.

COR. 14. But fince the forces NM, ML, when the body S is exceedingly diftant, are very nearly as the force SK and the ratio of PT to ST conjunctly; that is, if both the diftance PT, and the abfolute force of the body S be given, as ST^3 reciprocally; and fince thofe forces NM, ML are the caufes of all the errors and effects treated of in the foregoing corollaries; it is manifeft that all thofe effects, if the fyftem of bodies T and P continue as before, and only the diftance ST and the abfolute force of the body S be changed, will be very nearly in a ratio compounded of the direct ratio of the abfolute force of the body S, and the triplicate inverfe ratio of the diftance ST. Hence if the fyftem of bodies T and P revolve about a diftant body S, thofe forces NM, ML, and their effects, will be (by cor. 2 and 6, prop. 4) reciprocally in a duplicate ratio of the periodical time. And thence, alfo, if the magnitude of the body S be proportional to its abfolute force, thofe forces NM, ML, and their effects, will be directly as the cube of the apparent diameter of the diftant body S viewed from *T*, and fo *vice verfa*. For thefe ratios are the fame as the compounded ratio above-mentioned.

COR. 15. And becaufe if the orbits ESE and PAB, retaining their figure, proportions, and inclination to each other, fhould alter their magnitude; and the forces of the bodies S

and T ſhould either remain, or be changed in any given ratio; theſe forces (that is, the force of the body T, which obliges the body P to deflect from a rectilinear courſe into the orbit PAB, and the force of the body S, which cauſes the body P to deviate from that orbit) would act always in the ſame manner, and in the ſame proportion; it follows, that all the effects will be ſimilar and proportional, and the times of thoſe effects proportional alſo; that is, that all the linear errors will be as the diameters of the orbits, the angular errors the ſame as before; and the times of ſimilar linear errors, or equal angular errors, as the periodical times of the orbits.

COR. 16. Therefore if the figures of the orbits and their inclination to each other be given, and the magnitudes, forces, and diſtances of the bodies be any how changed, we may, from the errors and times of thoſe errors in one caſe, collect very nearly the errors and times of the errors in any other caſe. But this may be done more expeditiouſly by the following method. The forces NM, ML, other things remaining unaltered, are as the radius TP; and their periodical effects (by cor. 2, lem. 10) are as the forces and the ſquare of the periodical time of the body P conjunctly. Theſe are the linear errors of the body P; and hence the angular errors as they appear from the centre T (that is, the motion of the apſides and of the nodes, and all the apparent errors as to longitude and latitude) are in each revolution of the body P as the ſquare of the time of the revolution, very nearly. Let theſe ratios be compounded with the ratios in cor. 14, and in any ſyſtem of bodies T, P, S, where P revolves about T very near to it, and T revolves about S at a great diſtance, the angular errors of the body P, obſerved from the centre T, will be in each revolution of the body P as the ſquare of the periodical time of the body P directly, and the ſquare of the periodical time of the body T inverſely. And therefore the mean motion of the line of the apſides will be in a given ratio to the mean motion of the nodes; and both thoſe motions will be as the periodical time of the body P directly, and the ſquare of the periodical time of the body T inverſely. The

increafe or diminution of the eccentricity and inclination of the orbit PAB makes no fenfible variation in the motions of the apfides and nodes, unlefs that increafe or diminution be very great indeed.

COR. 17. Since the line LM becomes fometimes greater and fometimes lefs than the radius PT, let the mean quantity of the force LM be expreffed by that radius PT; and then that mean force will be to the mean force SK or SN (which may be alfo expreffed by ST) as the length PT to the length ST. But the mean force SN or ST, by which the body T is retained in the orbit it defcribes about S, is to the force with which the body P is retained in its orbit about T in a ratio compounded of the ratio of the radius ST to the radius PT, and the duplicate ratio of the periodical time of the body P about T to the periodical time of the body T about S. And, *ex æquo*, the mean force LM is to the force by which the body P is retained in its orbit about T (or by which the fame body P might revolve at the diftance PT in the fame periodical time about any immovable point T) in the fame duplicate ratio of the periodical times. The periodical times therefore being given, together with the diftance PT, the mean force LM is alfo given; and that force being given, there is given alfo the force MN, very nearly, by the analogy of the lines PS and MN.

COR. 18. By the fame laws by which the body P revolves about the body T, let us fuppofe many fluid bodies to move round T at equal diftances from it; and to be fo numerous, that they may all become contiguous to each other, fo as to form a fluid annulus, or ring, of a round figure, and concentrical to the body T; and the feveral parts of this annulus, performing their motions by the fame law as the body P, will draw nearer to the body T, and move fwifter in the conjunction and oppofition of themfelves and the body S, than in the quadratures. And the nodes of this annulus, or its interfections with the plane of the orbit of the body S or T, will reft at the fyzygies; but out of the fyzygies they will be carried backward, or *in antecedentia;* with the greateft fwiftnefs in the quadratures, and more flowly in other places. The inclina-

tion of this annulus also will vary, and its axis will oscillate each revolution, and when the revolution is completed will return to its former situation, except only that it will be carried round a little by the præcession of the nodes.

COR. 19. Suppose now the sphærical body T, consisting of some matter not fluid, to be enlarged, and to extend itself on every side as far as that annulus, and that a channel were cut all round its circumference containing water; and that this sphere revolves uniformly about its own axis in the same periodical time. This water being accelerated and retarded by turns (as in the last corollary), will be swifter at the syzygies, and slower at the quadratures, than the surface of the globe, and so will ebb and flow in its channel after the manner of the sea. If the attraction of the body S were taken away, the water would acquire no motion of flux and reflux by revolving round the quiescent centre of the globe. The case is the same of a globe moving uniformly forwards in a right line, and in the mean time revolving about its centre (by cor. 5 of the laws of motion), and of a globe uniformly attracted from its rectilinear course (by cor. 6, of the same laws). But let the body S come to act upon it, and by its unequable attraction the water will receive this new motion; for there will be a stronger attraction upon that part of the water that is nearest to the body, and a weaker upon that part which is more remote. And the force LM will attract the water downwards at the quadratures, and depress it as far as the syzygies; and the force KL will attract it upwards in the syzygies, and withhold its descent, and make it rise as far as the quadratures; except only in so far as the motion of flux and reflux may be directed by the channel of the water, and be a little retarded by friction.

COR. 20. If, now, the annulus becomes hard, and the globe is diminished, the motion of flux and reflux will cease; but the oscillating motion of the inclination and the præcession of the nodes will remain. Let the globe have the same axis with the annulus, and perform its revolutions in the same times, and at its surface touch the annulus within, and adhere to it; then the globe partaking of the motion of the annulus, this whole

compages will oscillate, and the nodes will go backward, for the globe, as we shall shew presently, is perfectly indifferent to the receiving of all impressions. The greatest angle of the inclination of the annulus single is when the nodes are in the syzygies. Thence in the progress of the nodes to the quadratures, it endeavours to diminish its inclination, and by that endeavour impresses a motion upon the whole globe. The globe retains this motion impressed, till the annulus by a contrary endeavour destroys that motion, and impresses a new motion in a contrary direction. And by this means the greatest motion of the decreasing inclination happens when the nodes are in the quadratures, and the least angle of inclination in the octants after the quadratures; and, again, the greatest motion of reclination happens when the nodes are in the syzygies; and the greatest angle of reclination in the octants following. And the case is the same of a globe without this annulus, if it be a little higher or a little denser in the equatorial than in the polar regions; for the excess of that matter in the regions near the equator supplies the place of the annulus. And though we should suppose the centripetal force of this globe to be any how increased, so that all its parts were to tend downwards, as the parts of our earth gravitate to the centre, yet the phænomena of this and the preceding corollary would scarce be altered; except that the places of the greatest and least height of the water will be different; for the water is now no longer sustained and kept in its orbit by its centrifugal force, but by the channel in which it flows. And, besides, the force LM attracts the water downwards most in the quadratures, and the force KL or NM — LM attracts it upwards most in the syzygies. And these forces conjoined cease to attract the water downwards, and begin to attract it upwards in the octants before the syzygies; and cease to attract the water upwards, and begin to attract the water downwards in the octants after the syzygies. And thence the greatest height of the water may happen about the octants after the syzygies; and the least height about the octants after the quadratures; excepting only so far as the motion of ascent or descent impressed by these forces may by the *vis*

insita of the water continue a little longer, or be stopt a little sooner by impediments in its channel.

Cor. 21. For the same reason that redundant matter in the equatorial regions of a globe causes the nodes to go backwards, and therefore by the increase of that matter that retrogradation is increased, by the diminution is diminished and by the removal quite ceases; it follows, that, if more than that redundant matter be taken away, that is, if the globe be either more depressed, or of a more rare consistence near the equator than near the poles, there will arise a motion of the nodes *in consequentia.*

Cor. 22. And thence from the motion of the nodes is known the constitution of the globe. That is, if the globe retains unalterably the same poles, and the motion (of the nodes) be *in antecedentia,* there is a redundance of the matter near the equator; but if *in consequentia,* a deficiency. Suppose an uniform and exactly sphærical globe to be first at rest in a free space; then by some impulse made obliquely upon its superficies to be driven from its place, and to receive a motion partly circular and partly right forward. Because this globe is perfectly indifferent to all the axes that pass through its centre, nor has a greater propensity to one axis or to one situation of the axis than to any other, it is manifest that by its own force it will never change its axis, or the inclination of it. Let now this globe be impelled obliquely by a new impulse in the same part of its superficies as before; and since the effect of an impulse is not at all changed by its coming sooner or later, it is manifest that these two impulses, successively impressed, will produce the same motion as if they were impressed at the same time; that is, the same motion as if the globe had been impelled by a simple force compounded of them both (by cor. 2, of the laws), that is, a simple motion about an axis of a given inclination. And the case is the same if the second impulse were made upon any other place of the equator of the first motion; and also if the first impulse were made upon any place in the equator of the motion which would be generated by the second impulse alone; and therefore, also, when both impulses are made in any places what-

foever; for thefe impulfes will generate the fame circular motion as if they were impreffed together, and at once, in the place of the interfections of the equators of thofe motions, which would be generated by each of them feparately. Therefore, a homogeneous and perfect globe will not retain feveral diftinct motions, but will unite all thofe that are impreffed on it, and reduce them into one; revolving, as far as in it lies, always with a fimple and uniform motion about one fingle given axis, with an inclination perpetually invariable. And the inclination of the axis, or the velocity of the rotation, will not be changed by centripetal force. For if the globe be fuppofed to be divided into two hemifpheres, by any plane whatfoever paffing through its own centre, and the centre to which the force is directed, that force will always urge each hemifphere equally; and therefore will not incline the globe any way as to its motion round its own axis. But let there be added any where between the pole and the equator a heap of new matter like a mountain, and this, by its perpetual endeavour to recede from the centre of its motion, will difturb the motion of the globe, and caufe its poles to wander about its fuperficies, defcribing circles about themfelves and their oppofite points. Neither can this enormous evagation of the poles be corrected, unlefs by placing that mountain either in one of the poles; in which cafe, by cor. 21, the nodes of the equator will go forwards; or in the equatorial regions, in which cafe, by cor. 20, the nodes will go backwards; or, laftly, by adding on the other fide of the axis a new quantity of matter, by which the mountain may be balanced in its motion; and then the nodes will either go forwards or backwards, as the mountain and this newly added matter happen to be nearer to the pole or to the equator.

PROPOSITION LXVII. THEOREM XXVII.

The fame laws of attraction being fuppofed, I fay, that the exterior body S *does, by radii drawn to the point* O, *the common centre of gravity of the interior bodies* P *and* T, *defcribe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipfis having*

its focus in that centre, than it can describe round the innermost and greatest body T *by radii drawn to that body.*

For the attractions of the body S (Pl. 21, Fig. 3) towards T and P compose its absolute attraction, which is more directed towards O, the common centre of gravity of the bodies T and P, than it is to the greatest body T; and which is more in a reciprocal proportion to the square of the distance SO, than it is to the square of the distance ST; as will easily appear by a little consideration.

PROPOSITION LXVIII. THEOREM XXVIII.

The same laws of attraction supposed, I say, that the exterior body S *will, by radii drawn to* O, *the common centre of gravity of the interior bodies* P *and* T, *describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipsis having its focus in that centre, if the innermost and greatest body be agitated by these attractions as well as the rest, than it would do if that body were either at rest as not attracted, or were much more or much less attracted, or much more or much less agitated.*

This may be demonstrated after the same manner as prop. 66, but by a more prolix reasoning, which I therefore pass over. It will be sufficient to consider it after this manner. From the demonstration of the last proposition it is plain, that the centre, towards which the body S is urged by the two forces conjunctly, is very near to the common centre of gravity of those two other bodies. If this centre were to coincide with that common centre, and moreover the common centre of gravity of all the three bodies were at rest, the body S on one side, and the common centre of gravity of the other two bodies on the other side, would describe true ellipses about that quiescent common centre. This appears from cor. 2, prop. 58, compared with what was demonstrated in prop. 64 and 65. Now this accurate elliptical motion will be disturbed a little by the distance of the centre of the two bodies from the centre towards which the third body S is attracted. Let there be added, moreover, a motion to the common centre of the three, and the perturbation will be increased yet more.

Therefore the perturbation is leaſt when the common centre of the three bodies is at reſt; that is, when the innermoſt and greateſt body T is attracted according to the ſame law as the reſt are; and is always greateſt when the common centre of the three, by the diminution of the motion of the body T, begins to be moved, and is more and more agitated.

Cor. And hence if more leſſer bodies revolve about the great one, it may eaſily be inferred that the orbits deſcribed will approach nearer to ellipſes; and the deſcriptions of areas will be more nearly equable, if all the bodies mutually attract and agitate each other with accelerative forces that are as their abſolute forces directly, and the ſquares of the diſtances inverſely; and if the focus of each orbit be placed in the common centre of gravity of all the interior bodies (that is, if the focus of the firſt and innermoſt orbit be placed in the centre of gravity of the greateſt and innermoſt body; the focus of the ſecond orbit in the common centre of gravity of the two innermoſt bodies; the focus of the third orbit in the common centre of gravity of the three innermoſt; and ſo on), than if the innermoſt body were at reſt, and was made the common focus of all the orbits.

PROPOSITION LXIX. THEOREM XXIX.

In a ſyſtem of ſeveral bodies A, B, C, D, *&c. if any one of thoſe bodies, as* A, *attract all the reſt,* B, C, D, *&c. with accelerative forces that are reciprocally as the ſquares of the diſtances from the attracting body; and another body, as* B, *attracts alſo the reſt,* A, C, D, *&c. with forces that are reciprocally as the ſquares of the diſtances from the attracting body; the abſolute forces of the attracting bodies* A *and* B *will be to each other as thoſe very bodies* A *and* B *to which thoſe forces belong.*

For the accelerative attractions of all the bodies B, C, D, towards A, are by the ſuppoſition equal to each other at equal diſtances; and in like manner the accelerative attractions of all the bodies towards B are alſo equal to each other at equal diſtances. But the abſolute attractive force of the body A is to the abſolute attractive force of the body B as the accelerative attraction of all the bodies towards A to the accelerative

attraction of all the bodies towards B at equal distances; and so is also the accelerative attraction of the body B towards A to the accelerative attraction of the body A towards B. But the accelerative attraction of the body B towards A is to the accelerative attraction of the body A towards B as the mass of the body A to the mass of the body B; because the motive forces which (by the 2d, 7th, and 8th definition) are as the accelerative forces and the bodies attracted conjunctly are here equal to one another by the third law. Therefore the absolute attractive force of the body A is to the absolute attractive force of the body B as the mass of the body A to the mass of the body B. Q.E.D.

COR. 1. Therefore if each of the bodies of the system A, B, C, D, &c. does singly attract all the rest with accelerative forces that are reciprocally as the squares of the distances from the attracting body, the absolute forces of all those bodies will be to each other as the bodies themselves.

COR. 2. By a like reasoning, if each of the bodies of the system A, B, C, D, &c. do singly attract all the rest with accelerative forces, which are either reciprocally or directly in the ratio of any power whatever of the distances from the attracting body; or which are defined by the distances from each of the attracting bodies according to any common law; it is plain that the absolute forces of those bodies are as the bodies themselves.

COR. 3. In a system of bodies whose forces decrease in the duplicate ratio of the distances, if the lesser revolve about one very great one in ellipses, having their common focus in the centre of that great body, and of a figure exceedingly accurate; and moreover by radii drawn to that great body describe areas proportional to the times exactly; the absolute forces of those bodies to each other will be either accurately or very nearly in the ratio of the bodies. And so on the contrary, This appears from cor. of prop. 68, compared with the first corollary of this prop.

SCHOLIUM.

These propositions naturally lead us to the analogy there is between centripetal forces, and the central bodies to which

those forces used to be directed; for it is reasonable to suppose that forces which are directed to bodies should depend upon the nature and quantity of those bodies, as we see they do in magnetical experiments. And when such cases occur, we are to compute the attractions of the bodies by assigning to each of their particles its proper force, and then collecting the sum of them all. I here use the word attraction in general for any endeavour, of what kind soever, made by bodies to approach to each other; whether that endeavour arise from the action of the bodies themselves, as tending mutually to or agitating each other by spirits emitted; or whether it arises from the action of the æther or of the air, or of any medium whatsoever, whether corporeal or incorporeal, any how impelling bodies placed therein towards each other. In the same general sense I use the word impulse, not defining in this treatise the species or physical qualities of forces, but investigating the quantities and mathematical proportions of them; as I observed before in the definitions. In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, when we enter upon physics, we compare those proportions with the phænomena of Nature, that we may know what conditions of those forces answer to the several kinds of attractive bodies. And this preparation being made, we argue more safely concerning the physical species, causes, and proportions of the forces. Let us see, then, with what forces sphærical bodies consisting of particles endued with attractive powers in the manner above spoken of must act mutually upon one another; and what kind of motions will follow from thence.

SECTION XII.

Of the attractive forces of sphærical bodies.

PROPOSITION LXX. THEOREM XXX.

If to every point of a sphærical surface there tend equal centripetal forces decreasing in the duplicate ratio of the distances from those points; I say, that a corpuscle placed within that superficies will not be attracted by those forces any way.

Let HIKL (Pl. 21, Fig. 4), be that sphærical superficies, and P a corpuscle placed within. Through P let there be drawn to this superficies the two lines HK, IL, intercepting very small arcs HI, KL; and because (by cor. 3, lem. 7) the triangles HPI, LPK are alike, those arcs will be proportional to the distances HP, LP; and any particles at HI and KL of the sphærical superficies, terminated by right lines passing through P, will be in the duplicate ratio of those distances. Therefore the forces of these particles exerted upon the body P are equal between themselves. For the forces are as the particles directly, and the squares of the distances inversely. And these two ratios compose the ratio of equality. The attractions, therefore, being made equally towards contrary parts, destroy each other. And by a like reasoning all the attractions through the whole sphærical superficies are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q.E.D.

PROPOSITION LXXI. THEOREM XXXI.

The same things supposed as above, I say, that a corpuscle placed without the sphærical superficies is attracted towards the centre of the sphere with a force reciprocally proportional to the square of its distance from that centre.

Let AHKB, ahkb (Pl. 21, Fig. 5), be two equal sphærical superficies described about the centres S, s; their diameters AB, ab; and let P and p be two corpuscles situate without the spheres in those diameters produced. Let there be drawn from the corpuscles the lines PHK, PIL, phk, pil, cutting off from the great circles AHB, ahb, the equal arcs HK, hk, IL, il; and to those lines let fall the perpendiculars SD, sd, SE, se, IR, ir; of which let SD, sd, cut PL, pl, in F and f. Let fall also to the diameters the perpendiculars IQ, iq. Let now the angles DPE, dpe vanish; and because DS and ds, ES and es are equal, the lines PE, PF, and pe, pf, and the lineolæ DF, df may be taken for equal; because their last ratio, when the angles DPE, dpe vanish together, is the ratio of equality. These things then supposed, it will be, as PI to PF so is RI to DF, and as pf to pi so is df or DF to ri; and, *ex æquo*, as PI × pf to PF × pi so is RI to ri, that is (by

cor. 3, lem. 7), so is the arc IH to the arc ih. Again, PI is to PS as IQ to SE, and ps to pi as se or SE to iq; and, *ex æquo*, PI × ps to PS × pi as IQ to iq. And compounding the ratios $PI^2 \times pf \times ps$ is to $pi^2 \times PF \times PS$, as IH × IQ to ih × iq; that is, as the circular superficies which is described by the arc IH, as the semi-circle AKB revolves about the diameter AB, is to the circular superficies described by the arc ih as the semi-circle akb revolves about the diameter ab. And the forces with which these superficies attract the corpuscles P and p in the direction of lines tending to those superficies are by the hypothesis as the superficies themselves directly, and the squares of the distances of the superficies from those corpuscles inversely; that is, as pf × ps to PF × PS. And these forces again are to the oblique parts of them which (by the resolution of forces as in cor. 2, of the laws) tend to the centres in the directions of the lines PS, ps, as PI to PQ, and pi to pq; that is (because of the like triangles PIQ and PSF, piq and psf), as PS to PF and ps to pf. Thence, *ex æquo*, the attraction of the corpuscle P towards S is to the attraction of the corpuscle p towards s as $\frac{PF \times pf \times ps}{PS}$ is to $\frac{pf \times PF \times PS}{ps}$, that is, as ps^2 to PS^2. And, by a like reasoning, the forces with which the superficies described by the revolution of the arcs KL, kl attract those corpuscles, will be as ps^2 to PS^2. And in the same ratio will be the forces of all the circular superficies into which each of the sphærical superficies may be divided by taking sd always equal to SD, and se equal to SE. And therefore, by composition, the forces of the entire sphærical superficies exerted upon those corpuscles will be in the same ratio. Q.E.D.

PROPOSITION LXXII. THEOREM XXXII.

If to the several points of a sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from those points; and there be given both the density of the sphere and the ratio of the diameter of the sphere to the distance of the corpuscle from its centre; I say, that the force with which the corpuscle is attracted is proportional to the semi-diameter of the sphere.

For conceive two corpuscles to be severally attracted by two spheres, one by one, the other by the other, and their distances from the centres of the spheres to be proportional to the diameters of the spheres respectively; and the spheres to be resolved into like particles, disposed in a like situation to the corpuscles. Then the attractions of one corpuscle towards the several particles of one sphere will be to the attractions of the other towards as many analogous particles of the other sphere in a ratio compounded of the ratio of the particles directly, and the duplicate ratio of the distances inversely. But the particles are as the spheres, that is, in a triplicate ratio of the diameters, and the distances are as the diameters; and the first ratio directly with the last ratio taken twice inversely, becomes the ratio of diameter to diameter. Q.E.D.

Cor. 1. Hence if corpuscles revolve in circles about spheres composed of matter equally attracting, and the distances from the centres of the spheres be proportional to their diameters, the periodic times will be equal.

Cor. 2. And, *vice versa*, if the periodic times are equal, the distances will be proportional to the diameters. These two corollaries appear from cor. 3, prop. 4.

Cor. 3. If to the several points of any two solids whatever, of like figure and equal density, there tend equal centripetal forces decreasing in a duplicate ratio of the distances from those points, the forces with which corpuscles placed in a like situation to those two solids will be attracted by them will be to each other as the diameters of the solids.

PROPOSITION LXXIII. THEOREM XXXIII.

If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that a corpuscle placed within the sphere is attracted by a force proportional to its distance from the centre.

In the sphere ABCD (Pl. 21, Fig. 6), described about the centre S, let there be placed the corpuscle P; and about the same centre S, with the interval SP, conceive described an interior sphere PEQF. It is plain (by prop. 70) that the

concentric ſphærical ſuperficies, of which the difference AEBF of the ſpheres is compoſed, have no effect at all upon the body P, their attractions being deſtroyed by contrary attractions. There remains, therefore, only the attraction of the interior ſphere PEQF. And (by prop. 72) this is as the diſtance PS. Q.E.D.

SCHOLIUM.

By the ſuperficies of which I here imagine the ſolids compoſed, I do not mean ſuperficies purely mathematical, but orbs ſo extremely thin, that their thickneſs is as nothing; that is, the evaneſcent orbs of which the ſphere will at laſt conſiſt, when the number of the orbs is increaſed, and their thickneſs diminiſhed without end. In like manner, by the points of which lines, ſurfaces, and ſolids are ſaid to be compoſed, are to be underſtood equal particles, whoſe magnitude is perfectly inconſiderable.

PROPOSITION LXXIV. THEOREM XXXIV.

The ſame things ſuppoſed, I ſay, that a corpuſcle ſituate without the ſphere is attracted with a force reciprocally proportional to the ſquare of its diſtance from the centre.

For ſuppoſe the ſphere to be divided into innumerable concentric ſphærical ſuperficies, and the attractions of the corpuſcle ariſing from the ſeveral ſuperficies will be reciprocally proportional to the ſquare of the diſtance of the corpuſcle from the centre of the ſphere (by prop. 71). And, by compoſition, the ſum of thoſe attractions, that is, the attraction of the corpuſcle towards the entire ſphere, will be in the ſame ratio. Q.E.D.

Cor. 1. Hence the attractions of homogeneous ſpheres at equal diſtances from the centres will be as the ſpheres themſelves. For (by prop. 72) if the diſtances be proportional to the diameters of the ſpheres, the forces will be as the diameters. Let the greater diſtance be diminiſhed in that ratio; and the diſtances now being equal, the attraction will be increaſed in the duplicate of that ratio; and therefore will be to the other attraction in the triplicate of that ratio; that is, in the ratio of the ſpheres.

Fig. 1.

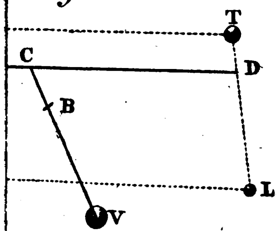


Fig. 2.

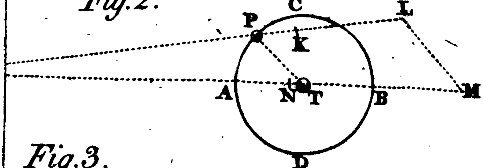


Fig. 3.

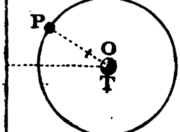


Fig. 4.

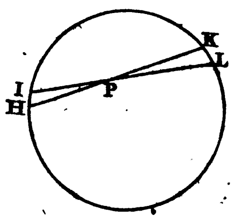


Fig. 5.

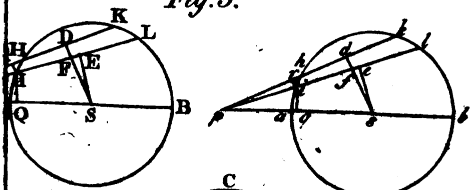
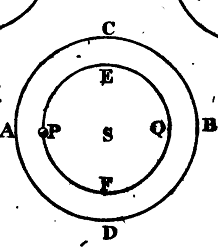


Fig. 6.



W. Newman. Sc 27 Widegate Street.

Cor. 2. At any diſtances whatever the attractions are as the ſpheres applied to the ſquares of the diſtances.

Cor. 3. If a corpuſcle placed without an homogeneous ſphere is attracted by a force reciprocally proportional to the ſquare of its diſtance from the centre, and the ſphere conſiſts of attractive particles, the force of every particle will decreaſe in a duplicate ratio of the diſtance from each particle.

PROPOSITION LXXV. THEOREM XXXV.

If to the ſeveral points of a given ſphere there tend equal centripetal forces decreaſing in a duplicate ratio of the diſtances from the points; I ſay, that another ſimilar ſphere will be attracted by it with a force reciprocally proportional to the ſquare of the diſtance of the centres.

For the attraction of every particle is reciprocally as the ſquare of its diſtance from the centre of the attracting ſphere (by prop. 74), and is therefore the ſame as if that whole attracting force iſſued from one ſingle corpuſcle placed in the centre of this ſphere. But this attraction is as great as on the other hand the attraction of the ſame corpuſcle would be, if that were itſelf attracted by the ſeveral particles of the attracted ſphere with the ſame force with which they are attracted by it. But that attraction of the corpuſcle would be (by prop. 74) reciprocally proportional to the ſquare of its diſtance from the centre of the ſphere; therefore the attraction of the ſphere, equal thereto, is alſo in the ſame ratio. Q.E.D.

Cor. 1. The attractions of ſpheres towards other homogeneous ſpheres are as the attracting ſpheres applied to the ſquares of the diſtances of their centres from the centres of thoſe which they attract.

Cor. 2. The caſe is the ſame when the attracted ſphere does alſo attract. For the ſeveral points of the one attract the ſeveral points of the other with the ſame force with which they themſelves are attracted by the others again; and therefore ſince in all attractions (by law 3) the attracted and attracting point are both equally acted on, the force will be doubled by their mutual attractions, the proportions remaining.

Cor. 3. Thoſe ſeveral truths demonſtrated above concerning the motion of bodies about the focus of the conic ſec-

tions will take place when an attracting sphere is placed in the focus, and the bodies move without the sphere.

COR. 4. Those things which were demonstrated before of the motion of bodies about the centre of the conic sections take place when the motions are performed within the sphere.

PROPOSITION LXXVI. THEOREM XXXVI.

If spheres be however dissimilar (as to density of matter and attractive force) in the same ratio onward from the centre to the circumference; but every where similar, at every given distance from the centre, on all sides round about; and the attractive force of every point decreases in the duplicate ratio of the distance of the body attracted; I say, that the whole force with which one of these spheres attracts the other will be reciprocally proportional to the square of the distance of the centres.

Imagine several concentric similar spheres, AB, CD, EF, &c. (Pl. 22, Fig. 1), the innermost of which added to the outermost may compose a matter more dense towards the centre, or subducted from them may leave the same more lax and rare. Then, by prop. 75, these spheres will attract other similar concentric spheres GH, IK, LM, &c. each the other, with forces reciprocally proportional to the square of the distance SP. And, by composition or division, the sum of all those forces, or the excess of any of them above the others; that is, the entire force with which the whole sphere AB (composed of any concentric spheres or of their differences) will attract the whole sphere GH (composed of any concentric spheres or their differences) in the same ratio. Let the number of the concentric spheres be increased *in infinitum*, so that the density of the matter together with the attractive force may, in the progress from the circumference to the centre, increase or decrease according to any given law; and by the addition of matter not attractive, let the deficient density be supplied, that so the spheres may acquire any form desired; and the force with which one of these attracts the other will be still, by the former reasoning, in the same ratio of the square of the distance inversely. Q.E.D.

Cor. 1. Hence if many spheres of this kind, similar in all respects, attract each other mutually, the accelerative attractions of each to each, at any equal distances of the centres, will be as the attracting spheres.

Cor. 2. And at any unequal distances, as the attracting spheres applied to the squares of the distances between the centres.

Cor. 3. The motive attractions, or the weights of the spheres towards one another, will be at equal distances of the centres as the attracting and attracted spheres conjunctly; that is, as the products arising from multiplying the spheres into each other.

Cor. 4. And at unequal distances, as those products directly, and the squares of the distances between the centres inversely.

Cor. 5. These proportions take place also when the attraction arises from the attractive virtue of both spheres mutually exerted upon each other. For the attraction is only doubled by the conjunction of the forces, the proportions remaining as before.

Cor. 6. If spheres of this kind revolve about others at rest, each about each; and the distances between the centres of the quiescent and revolving bodies are proportional to the diameters of the quiescent bodies; the periodic times will be equal.

Cor. 7. And, again, if the periodic times are equal, the distances will be proportional to the diameters.

Cor. 8. All those truths above demonstrated, relating to the motions of bodies about the foci of conic sections, will take place when an attracting sphere, of any form and condition like that above described, is placed in the focus.

Cor. 9. And also when the revolving bodies are also attracting spheres of any condition like that above described.

PROPOSITION LXXVII. THEOREM XXXVII.

If to the several points of spheres there tend centripetal forces proportional to the distances of the points from the attracted bodies; I say, that the compounded force with which two

ſpheres attract each other mutually is as the diſtance between the centres of the ſpheres.

CASE 1. Let AEBF (Pl. 22, Fig. 2) be a ſphere; S its centre; P a corpuſcle attracted; PASB the axis of the ſphere paſſing through the centre of the corpuſcle; EF, ef two planes cutting the ſphere, and perpendicular to the axis, and equi-diſtant, one on one ſide, the other on the other, from the centre of the ſphere; G and g the interſections of the planes and the axis; and H any point in the plane EF. The centripetal force of the point H upon the corpuſcle P, exerted in the direction of the line PH, is as the diſtance PH; and (by cor. 2, of the laws) the ſame exerted in the direction of the line PG, or towards the centre S, is as the length PG. Therefore the force of all the points in the plane EF (that is, of that whole plane) by which the corpuſcle P is attracted towards the centre S is as the diſtance PG multiplied by the number of thoſe points, that is, as the ſolid contained under that plane EF and the diſtance PG. And in like manner the force of the plane ef, by which the corpuſcle P is attracted towards the centre S, is as that plane drawn into its diſtance Pg, or as the equal plane EF drawn into that diſtance Pg; and the ſum of the forces of both planes as the plane EF drawn into the ſum of the diſtances PG + Pg, that is, as that plane drawn into twice the diſtance PS of the centre and the corpuſcle; that is, as twice the plane EF drawn into the diſtance PS, or as the ſum of the equal planes EF + ef drawn into the ſame diſtance. And, by a like reaſoning, the forces of all the planes in the whole ſphere, equi-diſtant on each ſide from the centre of the ſphere, are as the ſum of thoſe planes drawn into the diſtance PS, that is, as the whole ſphere and the diſtance PS conjunctly. Q.E.D.

CASE 2. Let now the corpuſcle P attract the ſphere AEBF. And, by the ſame reaſoning, it will appear that the force with which the ſphere is attracted is as the diſtance PS. Q.E.D.

CASE 3. Imagine another ſphere compoſed of innumerable corpuſcles P; and becauſe the force with which every cor-

pufcle is attracted is as the diftance of the corpufcle from the centre of the firft fphere, and as the fame fphere conjunctly, and is therefore the fame as if it all proceeded from a fingle corpufcle fituate in the centre of the fphere, the entire force with which all the corpufcles in the fecond fphere are attracted, that is, with which that whole fphere is attracted, will be the fame as if that fphere were attracted by a force iffuing from a fingle corpufcle in the centre of the firft fphere; and is therefore proportional to the diftance between the centres of the fpheres. Q.E.D.

Case 4. Let the fpheres attract each other mutually, and the force will be doubled, but the proportion will remain. Q.E.D.

Case 5. Let the corpufcle p be placed within the fphere AEBF (Fig. 3); and becaufe the force of the plane ef upon the corpufcle is as the folid contained under that plane and the diftance pg; and the contrary force of the plane EF as the folid contained under that plane and the diftance pG; the force compounded of both will be as the difference of the folids, that is, as the fum of the equal planes drawn into half the difference of the diftances; that is, as that fum drawn into pS, the diftance of the corpufcle from the centre of the fphere. And, by a like reafoning, the attraction of all the planes EF, ef, throughout the whole fphere, that is, the attraction of the whole fphere, is conjunctly as the fum of all the planes, or as the whole fphere, and as pS, the diftance of the corpufcle from the centre of the fphere. Q.E.D.

Case 6. And if there be compofed a new fphere out of innumerable corpufcles fuch as p, fituate within the firft fphere AEBF, it may be proved, as before, that the attraction, whether fingle of one fphere towards the other, or mutual of both towards each other, will be as the diftance pS of the centres. Q.E.D.

PROPOSITION LXXVIII. THEOREM XXXVIII.

If fpheres in the progrefs from the centre to the circumference be however diffimilar and unequable, but fimilar on every fide round about at all given diftances from the centre; and

the attractive force of every point be as the distance of the attracted body; I say, that the entire force with which two spheres of this kind attract each other mutually is proportional to the distance between the centres of the spheres.

This is demonstrated from the foregoing proposition, in the same manner as the 76th proposition was demonstrated from the 75th.

Cor. Those things that were above demonstrated in prop. 10 and 64, of the motion of bodies round the centres of conic sections, take place when all the attractions are made by the force of sphærical bodies of the condition above described, and the attracted bodies are spheres of the same kind.

SCHOLIUM.

I have now explained the two principal cases of attractions; to wit, when the centripetal forces decrease in a duplicate ratio of the distances, or increase in a simple ratio of the distances, causing the bodies in both cases to revolve in conic sections, and composing sphærical bodies whose centripetal forces observe the same law of increase or decrease in the recess from the centre as the forces of the particles themselves do; which is very remarkable. It would be tedious to run over the other cases, whose conclusions are less elegant and important, so particularly as I have done these. I chuse rather to comprehend and determine them all by one general method as follows.

LEMMA XXIX.

If about the centre S (Pl. 22, Fig. 4) *there be described any circle as* AEB, *and about the centre* P *there be also described two circles* EF, ef, *cutting the first in* E *and* e, *and the line* PS *in* F *and* f; *and there be let fall to* PS *the perpendiculars* ED, ed; *I say, that, if the distance of the arcs* EF, ef *be supposed to be infinitely diminished, the last ratio of the evanescent line* Dd *to the evanescent line* Ff *is the same as that of the line* PE *to the line* PS.

For if the line Pe cut the arc EF in q; and the right line Ee, which coincides with the evanescent arc Ee, be produced,

and meet the right line PS in T; and there be let fall from S to PE the perpendicular SG; then, becaufe of the like triangles DTE, dTe, DES, it will be as Dd to Ee fo DT to TE, or DE to ES; and becaufe the triangles Eeq, ESG (by lem. 8, and cor. 3, lem. 7) are fimilar, it will be as Ee to eq or Ff fo ES to SG; and, *ex æquo*, as Dd to Ff fo DE to SG; that is (becaufe of the fimilar triangles PDE, PGS), fo is PE to PS. Q.E.D.

PROPOSITION LXXIX. THEOREM XXXIX.

Suppofe a fuperficies as EFfe (Pl. 22. Fig. 5) *to have its breadth infinitely diminifhed, and to be juft vanifhing; and that the fame fuperficies by its revolution round the axis* PS *defcribes a fphærical concavo-convex folid, to the feveral equal particles of which there tend equal centripetal forces; I fay, that the force with which that folid attracts a corpufcle fituate in* P *is in a ratio compounded of the ratio of the folid* $DE^2 \times Ff$ *and the ratio of the force with which the given particle in the place* Ff *would attract the fame corpufcle.*

For if we confider, firft, the force of the fphærical fuperficies FE which is generated by the revolution of the arc FE, and is cut any where, as in r, by the line de, the annular part of the fuperficies generated by the revolution of the arc rE will will be as the lineolæ Dd, the radius of the fphere PE remaining the fame; as *Archimedes* has demonftrated in his book of the fphere and cylinder. And the force of this fuperficies exerted in the direction of the lines PE or Pr fituate all round in the conical fuperficies, will be as this annular fuperficies itfelf; that is, as the lineolæ Dd, or, which is the fame, as the rectangle under the given radius PE of the fphere and the lineolæ Dd; but that force, exerted in the direction of the line PS tending to the centre S, will be lefs in the ratio of PD to PE, and therefore will be as $PD \times Dd$. Suppofe now the line DF to be divided into innumerable little equal particles, each of which call Dd, and then the fuperficies FE will be divided into fo many equal annuli, whofe forces will be as the fum of all the rectangles $PD \times Dd$, that is, as $\frac{1}{2}PF^2 - \frac{1}{2}PD^2$, and therefore as DE^2. Let now the fuperficies FE be drawn

into the altitude Ff; and the force of the solid EFfe exerted upon the corpuscle P will be as $DE^2 \times Ff$; that is, if the force be given which any given particle as Ff exerts upon the corpuscle P at the distance PF. But if that force be not given, the force of the solid EFfe will be as the solid $DE^2 \times Ff$ and that force not given, conjunctly. Q.E.D.

PROPOSITION LXXX. THEOREM XL.

If to the several equal parts of a sphere ABE (Pl. 22, Fig. 6) *described about the centre* S *there tend equal centripetal forces; and from the several points* D *in the axis of the sphere* AB *in which a corpuscle, as* P, *is placed, there be erected the perpendiculars* DE *meeting the sphere in* E, *and if in those perpendiculars the lengths* DN *be taken as the quantity* $\frac{DE^2 \times PS}{PE}$, *and as the force which a particle of the sphere situate in the axis exerts at the distance* PE *upon the corpuscle* P *conjunctly; I say, that the whole force with which the corpuscle* P *is attracted towards the sphere is as the area* ANB, *comprehended under the axis of the sphere* AB, *and the curve line* ANB, *the locus of the point* N.

For supposing the construction in the last lemma and theorem to stand, conceive the axis of the sphere AB to be divided into innumerable equal particles Dd, and the whole sphere to be divided into so many sphærical concavo-convex laminæ EFfe; and erect the perpendicular dn. By the last theorem, the force with which the laminæ EFfe attracts the corpuscle P is as $DE^2 \times Ff$ and the force of one particle exerted at the distance PE or PF, conjunctly. But (by the last lemma) Dd is to Ff as PE to PS, and therefore Ff is equal to $\frac{PS \times Dd}{PE}$; and $DE^2 \times Ff$ is equal to $Dd \times \frac{DE^2 \times PS}{PE}$; and therefore the force of the laminæ EFfe is as $Dd \times \frac{DE^2 \times PS}{PE}$ and the force of a particle exerted at the distance PF conjunctly; that is, by the supposition, as $DN \times Dd$, or as the evanescent area DNnd. Therefore the forces of all the laminæ exerted upon the corpuscle P are as all the areas DNnd, that is, the whole force of the sphere will be as the whole area ANB. Q.E.D.

COR. 1. Hence if the centripetal force tending to the several particles remain always the same at all distances, and DN be made as $\frac{DE^2 \times PS}{PE}$, the whole force with which the corpuscle is attracted by the sphere is as the area ANB.

COR. 2. If the centripetal force of the particles be reciprocally as the distance of the corpuscle attracted by it, and DN be made as $\frac{DE^2 \times PS}{PE^2}$, the force with which the corpuscle P is attracted by the whole sphere will be as the area ANB.

COR. 3. If the centripetal force of the particles be reciprocally as the cube of the distance of the corpuscle attracted by it, and DN be made as $\frac{DE^2 \times PS}{PE^4}$, the force with which the corpuscle is attracted by the whole sphere will be as the area ANB.

COR. 4. And universally if the centripetal force tending to the several particles of the sphere be supposed to be reciprocally as the quantity V; and DN be made as $\frac{DE^2 \times PS}{PE \times V}$; the force with which a corpuscle is attracted by the whole sphere will be as the area ANB.

PROPOSITION LXXXI. PROBLEM XLI.

The things remaining as above, it is required to measure the area ANB. (Pl. 23, Fig. 1.)

From the point P let there be drawn the right line PH touching the sphere in H; and to the axis PAB, letting fall the perpendicular HI, bisect PI in L; and (by prop. 12, book 2, elem.) PE^2 is equal to $PS^2 + SE^2 + 2PSD$. But because the triangles SPH, SHI are alike, SE^2 or SH^2 is equal to the rectangle PSI. Therefore PE^2 is equal to the rectangle contained under PS and PS + SI + 2SD; that is, under PS and 2LS + 2SD; that is, under PS and 2LD. Moreover DE^2 is equal to $SE^2 - SD^2$, or $SE^2 - LS^2 + 2SLD - LD^2$, that is, $2SLD - LD^2 - ALB$. For $LS^2 - SE^2$ or $LS^2 - SA^2$ (by prop. 6, book 2, elem.) is equal to the rectangle ALB. Therefore if instead of DE^2 we write $2SLD - LD^2 - ALB$, the quantity $\frac{DE^2 \times PS}{PE \times V}$, which (by cor. 4 of the foregoing prop.) is as the length of the ordinate DN,

will now resolve itself into three parts $\frac{2SLD \times PS}{PE \times V} - \frac{LD^2 \times PS}{PE \times V}$ $- \frac{ALB \times PS}{PE \times V}$; where if instead of V we write the *inverse* ratio of the centripetal force, and instead of PE the mean proportional between PS and 2LD, those three parts will become ordinates to so many curve lines, whose areas are discovered by the common methods. Q.E.D.

EXAMPLE 1. If the centripetal force tending to the several particles of the sphere be reciprocally as the distance; instead of V write PE the distance, then $2PS \times LD$ for PE^2; and DN will become as $SL - \frac{1}{2} LD - \frac{ALB}{2LD}$. Suppose DN equal to its double $2SL - LD - \frac{ALB}{LD}$; and $2SL$ the given part of the ordinate drawn into the length AB will describe the rectangular area $2SL \times AB$; and the indefinite *part LD*, drawn perpendicularly into the same length with a continued motion, in such sort as in its motion one way or another it may either by increasing or decreasing remain always equal to the length LD, will describe the area $\frac{LB^2 - LA^2}{2}$, that is, the area $SL \times AB$; which taken from the former area $2SL \times AB$, leaves the area $SL \times AB$. But the *third part* $\frac{ALB}{LD}$, drawn after the same manner with a continued motion perpendicularly into the same length, will describe the area of an hyperbola, which subducted from the area $SL \times AB$ will leave ANB the area sought. Whence arises this construction of the problem. At the points L, A, B (Fig. 2), erect the perpendiculars Ll, Aa, Bb; making Aa equal to LB, and Bb equal to LA. Making Ll and LB asymptotes, describe through the points a, b, the hyperbolic curve ab. *And the* chord ba being drawn, will inclose the area aba *equal to the* area sought ANB.

EXAMPLE 2. If the centripetal force tending to the several particles of the sphere be reciprocally as the cube of the distance, or (which is the same thing) as that cube applied to any given plane; write $\frac{PE^3}{2AS^2}$ for V, and $2PS \times LD$ for PE^2;

and DN will become as $\frac{SL \times AS^2}{PS \times LD} - \frac{AS^2}{2PS} - \frac{ALB \times AS^2}{2PS \times LD^2}$, that is (because PS, AS, SI are continually proportional), as $\frac{LSI}{LD} - \frac{1}{2}SI - \frac{ALB \times SI}{2LD^2}$. If we draw then these three parts into the length AB, the first $\frac{LSI}{LD}$ will generate the area of an hyperbola; the second $\frac{1}{2}SI$ the area $\frac{1}{2}AB \times SI$; the third $\frac{ALB \times SI}{2LD^2}$ the area $\frac{ALB \times SI}{2LA} - \frac{ALB \times SI}{2LB}$, that is, $\frac{1}{2}AB \times SI$. From the first subduct the sum of the second and third, and there will remain ANB, the area sought. Whence arises this construction of the problem. At the points L, A, S, B (Fig. 3), erect the perpendiculars Ll, Aa, Ss, Bb, of which suppose Ss equal to SI; and through the point s, to the asymptotes Ll, LB, describe the hyperbola asb meeting the perpendiculars Aa, Bb, in a and b; and the rectangle 2ASI, subducted from the hyperbolic area AasbB, will leave ANB the area sought.

Example 3. If the centripetal force tending to the several particles of the spheres decrease in a quadruplicate ratio of the distance from the particles; write $\frac{PE^4}{2AS^3}$ for V, then $\sqrt{2PS + LD}$ for PE, and DN will become as $\frac{SI^2 \times SL}{\sqrt{2SI}} \times \frac{1}{\sqrt{LD^3}} - \frac{SI^2}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD}} - \frac{SI^2 \times ALB}{2\sqrt{2SI}} \times \frac{1}{\sqrt{LD^5}}$. These three parts drawn into the length AB, produce so many areas, viz. $\frac{2SI^2 \times SL}{\sqrt{2SI}}$ into $\frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}$; $\frac{SI^2}{\sqrt{2SI}}$ into $\sqrt{LB} - \sqrt{LA}$; and $\frac{SI^2 \times ALB}{3\sqrt{2SI}}$ into $\frac{1}{\sqrt{LA^3}} - \frac{1}{\sqrt{LB^3}}$. And these after due reduction come forth $\frac{2SI^2 \times SL}{LI}$, SI^2, and $SI^2 + \frac{2SI^3}{3LI}$. And these, by subducting the last from the first, become $\frac{4SI^3}{3LI}$. Therefore the entire force with which the corpuscle P is attracted towards the centre of the

ſphere is as $\frac{SI^3}{PI}$, that is, reciprocally as $PS^3 \times PI$. Q.E.I.

By the ſame method one may determine the attraction of a corpuſcle ſituate within the ſphere, but more expeditiouſly by the following theorem.

PROPOSITION LXXXII. THEOREM XLI.

In a ſphere deſcribed about the centre S (Pl. 23, Fig. 4) *with the interval* SA, *if there be taken* SI, SA, SP *continually proportional; I ſay, that the attraction of a corpuſcle within the ſphere in any place* I *is to its attraction without the ſphere in the place* P *in a ratio compounded of the ſubduplicate ratio of* IS, PS, *the diſtances from the centre, and the ſubduplicate ratio of the centripetal forces tending to the centre in thoſe places* P *and* I.

As if the centripetal forces of the particles of the ſphere be reciprocally as the diſtances of the corpuſcle attracted by them; the force with which the corpuſcle ſituate in I is attracted by the entire ſphere will be to the force with which it is attracted in P in a ratio compounded of the ſubduplicate ratio of the diſtance SI to the diſtance SP, and the ſubduplicate ratio of the centripetal force in the place I ariſing from any particle in the centre to the centripetal force in the place P ariſing from the ſame particle in the centre; that is, in the ſubduplicate ratio of the diſtances SI, SP to each other reciprocally. Theſe two ſubduplicate ratios compoſe the ratio of equality, and therefore the attractions in I and P produced by the whole ſphere are equal. By the like calculation, if the forces of the particles of the ſphere are reciprocally in a duplicate ratio of the diſtance, it will be found that the attraction in I is to the attraction in P as the diſtance SP to the ſemidiameter SA of the ſphere. If thoſe forces are reciprocally in a triplicate ratio of the diſtances, the attractions in I and P will be to each other as SP^2 to SA^2; if in a quadruplicate ratio, as SP^3 to SA^3. Therefore ſince the attraction in P was found in this laſt caſe to be reciprocally as $PS^3 \times PI$, the attraction in I will be reciprocally as $SA^3 \times PI$, that is, becauſe SA^3 is given, reciprocally as PI. And the progreſſion is the ſame *in infinitum*. The demonſtration of this theorem is as follows:

The things remaining as above constructed, and a corpuscle being in any place P, the ordinate DN was found to be as $\frac{DE^2 \times PS}{PE \times V}$. Therefore if IE be drawn, that ordinate for any other place of the corpuscle, as I, will become (*mutatis mutandis*) as $\frac{DE^2 \times IS}{IE \times V}$. Suppose the centripetal forces flowing from any point of the sphere, as E, to be to each other at the distances IE and PE as PE^n to IE^n (where the number n denotes the index of the powers of PE and IE), and those ordinates will become as $\frac{DE^2 \times PS}{PE \times PE^n}$ and $\frac{DE^2 \times IS}{IE \times IE^n}$ whose ratio to each other is as $PS \times IE \times IE^n$ to $IS \times PE \times PE^n$. Because SI, SE, SP are in continued proportion, the triangles SPE, SEI are alike; and thence IE is to PE as IS to SE or SA. For the ratio of IE to PE write the ratio of IS to SA; and the ratio of the ordinates becomes that of $PS \times IE^n$ to $SA \times PE^n$. But the ratio of PS to SA is subduplicate of that of the distances PS, SI; and the ratio of IE^n to PE^n (because IE is to PE as IS to SA) is subduplicate of that of the forces at the distances PS, IS. Therefore the ordinates, and consequently the areas which the ordinates describe, and the attractions proportional to them, are in a ratio compounded of those subduplicate ratios. Q.E.D.

PROPOSITION LXXXIII. PROBLEM XLII.

To find the force with which a corpuscle placed in the centre of a sphere is attracted towards any segment of that sphere whatsoever.

Let P (Pl. 23, Fig. 5) be a body in the centre of that sphere, and RBSD a segment thereof contained under the plane RDS, and the sphærical superficies RBS. Let DB be cut in F by a sphærical superficies EFG described from the centre P, and let the segment be divided into the parts BREFGS, FEDG. Let us suppose that segment to be not a purely mathematical but a physical superficies, having some, but a perfectly inconsiderable, thickness. Let that thickness be called O, and (by what *Archimedes* has demonstrated) that superficies will be as $PF \times DF \times O$. Let us suppose besides

the attractive forces of the particles of the sphere to be reciprocally as that power of the distances, of which n is index; and the force with which the superficies EFG attracts the body P will be (by prop. 79) as $\frac{DE^2 \times O}{PF^n}$, that is, as $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^2 \times O}{PF^n}$. Let the perpendicular FN drawn into O be proportional to this quantity; and the curvilinear area BDI, which the ordinate FN, drawn through the length DB with a continued motion will describe, will be as the whole force with which the whole segment RBSD attracts the body P. Q.E.I.

PROPOSITION LXXXIV. PROBLEM XLIII.

To find the force with which a corpuscle, placed without the centre of a sphere in the axis of any segment, is attracted by that segment.

Let the body P placed in the axis ADB of the segment EBK (Pl. 23, Fig. 6) be attracted by that segment. About the centre P, with the interval PE, let the spherical superficies EFK be described; and let it divide the segment into two parts EBKFE and EFKDE. Find the force of the first of those parts by prop. 81, and the force of the latter part by prop. 83, and the sum of the forces will be the force of the whole segment EBKDE. Q.E.I.

SCHOLIUM.

The attractions of sphærical bodies being now explained, it comes next in order to treat of the laws of attraction in other bodies consisting in like manner of attractive particles; but to treat of them particularly is not necessary to my design. It will be sufficient to subjoin some general propositions relating to the forces of such bodies, and the motions thence arising, because the knowledge of these will be of some little use in philosophical enquiries.

SECTION XIII.

Of the attractive forces of bodies which are not of a sphærical figure.

PROPOSITION LXXXV. THEOREM XLII.

If a body be attracted by another, and its attraction be vastly stronger when it is contiguous to the attracting body than

when they are separated from one another by a very small interval; the forces of the particles of the attracting body decrease, in the recess of the body attracted, in more than a duplicate ratio of the distance of the particles.

For if the forces decrease in a duplicate ratio of the distances from the particles, the attraction towards a sphærical body being (by prop. 74) reciprocally as the square of the distance of the attracted body from the centre of the sphere, will not be sensibly increased by the contact, and it will be still less increased by it, if the attraction, in the recess of the body attracted, decreases in a still less proportion. The proposition, therefore, is evident concerning attractive spheres. And the case is the same of concave sphærical orbs attracting external bodies. And much more does it appear in orbs that attract bodies placed within them, because there the attractions diffused through the cavities of those orbs are (by prop. 70) destroyed by contrary attractions, and therefore have no effect even in the place of contact. Now if from these spheres and sphærical orbs we take away any parts remote from the place of contact, and add new parts any where at pleasure, we may change the figures of the attractive bodies at pleasure; but the parts added or taken away, being remote from the place of contact, will cause no remarkable excess of the attraction arising from the contact of the two bodies. Therefore the proposition holds good in bodies of all figures. Q.E.D.

PROPOSITION LXXXVI. THEOREM XLIII.

If the forces of the particles of which an attractive body is composed decrease, in the recess of the attracted body, in a triplicate or more than a triplicate ratio of the distance from the particles, the attraction will be vastly stronger in the point of contact than when the attracting and attracted bodies are separated from each other, though by never so small an interval.

For that the attraction is infinitely increased when the attracted corpuscle comes to touch an attracting sphere of this kind, appears, by the solution of problem 41, exhibited in the second and third examples. The same will also appear (by

comparing those examples and theorem 41 together) of attractions of bodies made towards concavo-convex orbs, whether the attracted bodies be placed without the orbs, or in the cavities within them. And by adding to or taking from those spheres and orbs any attractive matter any where without the place of contact, so that the attractive bodies may receive any assigned figure, the proposition will hold good of all bodies universally. Q.E.D.

PROPOSITION LXXXVII. THEOREM XLIV.

If two bodies similar to each other, and consisting of matter equally attractive, attract separately two corpuscles proportional to those bodies, and in a like situation to them, the accelerative attractions of the corpuscles towards the entire bodies will be as the accelerative attractions of the corpuscles towards particles of the bodies proportional to the wholes, and alike situated in them.

For if the bodies are divided into particles proportional to the wholes, and alike situated in them, it will be, as the attraction towards any particle of one of the bodies to the attraction towards the correspondent particle in the other body, so are the attractions towards the several particles of the first body to the attractions towards the several correspondent particles of the other body; and, by composition, so is the attraction towards the first whole body to the attraction towards the second whole body. Q.E.D.

Cor. 1. Therefore if, as the distances of the corpuscles attracted increase, the attractive forces of the particles decrease in the ratio of any power of the distances, the accelerative attractions towards the whole bodies will be as the bodies directly, and those powers of the distances inversely. As if the forces of the particles decrease in a duplicate ratio of the distances from the corpuscles attracted, and the bodies are as A^3 and B^3, and therefore both the cubic sides of the bodies, and the distance of the attracted corpuscles from the bodies, are as A and B; the accelerative attractions towards the bodies will be as $\frac{A^3}{A^2}$ and $\frac{B^3}{B^2}$, that is, as A and B the cubic sides of those bodies. If the forces of the particles decrease in a tri-

plicate ratio of the distances from the attracted corpuscles, the accelerative attractions towards the whole bodies will be as $\frac{A^3}{A^3}$ and $\frac{B^3}{B^3}$, that is, equal. If the forces decrease in a quadruplicate ratio, the attractions towards the bodies will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$, that is, reciprocally as the cubic sides A and B. And so in other cases.

Cor. 2. Hence, on the other hand, from the forces with which like bodies attract corpuscles similarly situated, may be collected the ratio of the decrease of the attractive forces of the particles as the attracted corpuscle recedes from them; if so be that decrease is directly or inversely in any ratio of the distances.

PROPOSITION LXXXVIII. THEOREM XLV.

If the attractive forces of the equal particles of any body be as the distance of the places from the particles, the force of the whole body will tend to its centre of gravity; and will be the same with the force of a globe, consisting of similar and equal matter, and having its centre in the centre of gravity.

Let the particles A, B (Pl. 23, Fig. 7), of the body RSTV attract any corpuscle Z with forces which, supposing the particles to be equal between themselves, are as the distances AZ, BZ; but, if they are supposed unequal, are as those particles and their distances AZ, BZ conjunctly, or (if I may so speak) as those particles drawn into their distances AZ, BZ respectively. And let those forces be expressed by the contents under A × AZ, and B × BZ. Join AB, and let it be cut in G, so that AG may be to BG as the particle B to the particle A; and G will be the common centre of gravity of the particles A and B. The force A × AZ will (by cor. 2, of the laws) be resolved into the forces A × GZ and A × AG; and the force B × BZ into the forces B × GZ and B × BG. Now the forces A × AG and B × BG, because A is proportional to B, and BG to AG, are equal, and therefore having contrary directions destroy one another. There remain then the forces A × GZ and B × GZ. These tend from Z towards

the centre G, and compose the force $\overline{A+B} \times GZ$; that is, the same force as if the attractive particles A and B were placed in their common centre of gravity G, composing there a little globe.

By the same reasoning, if there be added a third particle C, and the force of it be compounded with the force $\overline{A+B} \times GZ$ tending to the centre G, the force thence arising will tend to the common centre of gravity of that globe in G and of the particle C; that is, to the common centre of gravity of the three particles A, B, C; and will be the same as if that globe and the particle C were placed in that common centre composing a greater globe there; and so we may go on *in infinitum*. Therefore the whole force of all the particles of any body whatever RSTV, is the same as if that body, without removing its centre of gravity, were to put on the form of a globe. Q.E.D.

COR. Hence the motion of the attracted body Z will be the same as if the attracting body RSTV were sphærical; and therefore if that attracting body be either at rest, or proceed uniformly in a right line, the body attracted will move in an ellipsis having its centre in the centre of gravity of the attracting body.

PROPOSITION LXXXIX. THEOREM XLVI.

If there be several bodies consisting of equal particles whose forces are as the distances of the places from each, the force compounded of all the forces by which any corpuscle is attracted will tend to the common centre of gravity of the attracting bodies; and will be the same as if those attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe.

This is demonstrated after the same manner as the foregoing proposition.

COR. Therefore the motion of the attracted body will be the same as if the attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe. And, therefore, if the common centre of gravity of the attracting bodies be either at rest, or proceeds uniformly in a right line, the attracted body will move in an ellipsis

Fig. 1.

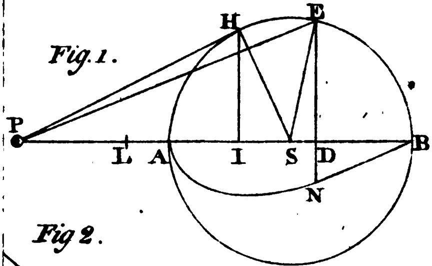
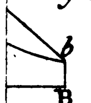


Fig 2.



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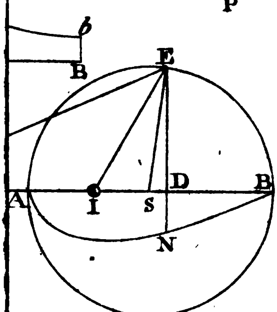


Fig 5.

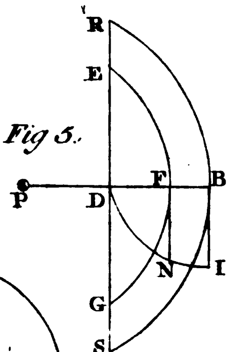
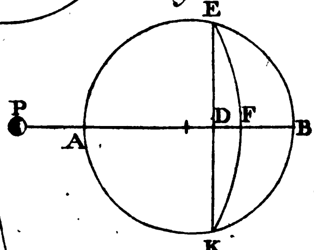


Fig 6.



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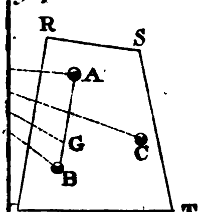

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having its centre in the common centre of gravity of the attracting bodies.

PROPOSITION XC. PROBLEM XLIV.

If to the several points of any circle there tend equal centripetal forces, increasing or decreasing in any ratio of the distances; it is required to find the force with which a corpuscle is attracted, that is, situate any where in a right line which stands at right angles to the plane of the circle at its centre.

Suppose a circle to be described about the centre A (Pl. 24, Fig. 1) with any interval AD in a plane to which the right line AP is perpendicular; and let it be required to find the force with which a corpuscle P is attracted towards the same. From any point E of the circle, to the attracted corpuscle P, let there be drawn the right line PE. In the right line PA take PF equal to PE, and make a perpendicular FK, erected at F, to be as the force with which the point E attracts the corpuscle P. And let the curve line IKL be the locus of the point K. Let that curve meet the plane of the circle in L. In PA take PH equal to PD, and erect the perpendicular HI meeting that curve in I; and the attraction of the corpuscle P towards the circle will be as the area AHIL drawn into the altitude AP. Q.E.I.

For let there be taken in AE a very small line Ee. Join Pe, and in PE, PA take PC, Pf equal to Pe. And because the force with which any point E of the annulus described about the centre A with the interval AE in the aforesaid plane attracts to itself the body P, is supposed to be as FK; and, therefore, the force with which that point attracts the body P towards A is as $\frac{AP \times FK}{PE}$; and the force with which the whole annulus attracts the body P towards A is as the annulus and $\frac{AP \times FK}{PE}$ conjunctly; and that annulus also is as the rectangle under the radius AE and the breadth Ee, and this rectangle (because PE and AE, Ee and CE are proportional) is equal to the rectangle PE × CE or PE × Ff;

the force with which that annulus attracts the body P towards A will be as PE × Ff and $\frac{AP \times FK}{PE}$ conjunctly; *that* is, as the content under Ff × FK × AP, or as the area *FKkf* drawn into AP. And therefore the ſum of the forces with which all the annuli, in the circle deſcribed about the centre A with the interval AD, attract the body P towards A, is as the whole area AHIKL drawn into AP. Q.E.D.

Cor. 1. Hence if the forces of the points decreaſe in the duplicate ratio of the diſtances, that is, if FK be as $\frac{1}{PF^2}$, and therefore the area AHIKL as $\frac{1}{PA} - \frac{1}{PH}$; the attraction of the corpuſcle P towards the circle will be as $1 - \frac{PA}{PH}$; that is, as $\frac{AH}{PH}$.

Cor. 2. And univerſally if the forces of the points at the diſtances D be reciprocally as any power D^n of the diſtances; that is, if FK be as $\frac{1}{D^n}$, and therefore the area AHIKL as $\frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}$; the attraction of the corpuſcle P towards the circle will be as $\frac{1}{PA^{n-2}} - \frac{PA}{PH^{n-1}}$.

Cor. 3. And if the diameter of the circle be increaſed *in infinitum,* and the number n be greater than unity; the attraction of the corpuſcle P towards the whole infinite plane will be reciprocally as PA^{n-2}, becauſe the other term $\frac{PA}{PH^{n-1}}$ vaniſhes.

PROPOSITION XCI. PROBLEM XLV.

To find the attraction of a corpuſcle ſituate in the axis of a round ſolid, to whoſe ſeveral points there tend equal centripetal forces decreaſing in any ratio of the diſtances whatſoever.

Let the corpuſcle P (Pl. 24, Fig. 2) ſituate in the axis AB of the ſolid DECG, be attracted towards that ſolid. Let the

ſolid be cut by any circle as RFS, perpendicular to the axis; and in its ſemi-diameter FS, in any plane PALKB paſſing through the axis, let there be taken (by prop. 90) the length FK proportional to the force with which the corpuſcle P is attracted towards that circle. Let the locus of the point K be the curve line LKI, meeting the planes of the outermoſt circles AL and BI in L and I; and the attraction of the corpuſcle P towards the ſolid will be as the area LABI. Q.E.I.

Cor. 1. Hence if the ſolid be a cylinder deſcribed by the parallelogram ADEB (Pl. 24, Fig. 3) revolved about the axis AB, and the centripetal forces tending to the ſeveral points be reciprocally as the ſquares of the diſtances from the points; the attraction of the corpuſcle P towards this cylinder will be as AB — PE + PD. For the ordinate FK (by cor. 1, prop. 90) will be as $1 - \frac{PF}{PR}$. The part 1 of this quantity, drawn into the length AB, deſcribes the area $1 \times AB$; and the other part $\frac{PF}{PR}$, drawn into the length PB, deſcribes the area 1 into $\overline{PE - AD}$ (as may be eaſily ſhewn from the quadrature of the curve LKI); and, in like manner, the ſame part drawn into the length PA deſcribes the area 1 into $\overline{PD - AD}$, and drawn into AB, the difference of PB and PA, deſcribes 1 into $\overline{PE - PD}$, the difference of the areas. From the firſt content $1 \times AB$ take away the laſt content 1 into $\overline{PE - PD}$, and there will remain the area LABI equal to 1 into $\overline{AB - PE + PD}$. Therefore the force being proportional to this area, is as AB — PE + PD.

Cor. 2. Hence alſo is known the force by which a ſpheroid AGBC (Pl. 24, Fig. 4) attracts any body P ſituate externally in its axis AB. Let NKRM be a conic ſection whoſe ordinate ER perpendicular to PE may be always equal to the length of the line PD, continually drawn to the point D in which that ordinate cuts the ſpheroid. From the vertices A, B, of the ſpheroid, let there be erected to its axis AB the perpendiculars AK, BM, reſpectively equal to AP, BP, and therefore meeting the conic ſection in K and M; and join KM cutting

off from it the ſegment KMRK. Let S be the centre of the ſpheroid, and SC its greateſt ſemi-diameter; and the force with which the ſpheroid attracts the body P will be to the force with which a ſphere deſcribed with the diameter AB attracts the ſame body, as $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ is to $\frac{AS^3}{3PS^2}$. And by a calculation founded on the ſame principles may be found the forces of the ſegments of the ſpheroid.

COR. 3. If the corpuſcle be placed within the ſpheroid and in its axis, the attraction will be as its diſtance from the centre. This may be eaſily collected from the following reaſoning, whether the particle be in the axis or in any other given diameter. Let AGOF (Pl. 24, Fig. 5) be an attracting ſpheroid, S its centre, and P the body attracted. Through the body P let there be drawn the ſemi-diameter SPA, and two right lines DE, FG meeting the ſpheroid in D and E, F and G; and let PCM, HLN be the ſuperficies of two interior ſpheroids ſimilar and concentrical to the exterior, the firſt of which paſſes through the body P, and cuts the right lines DE, FG in B and C; and the latter cuts the ſame right lines in H and I, K and L. Let the ſpheroids have all one common axis, and the parts of the right lines intercepted on both ſides DP and BE, FP and CG, DH and IE, FK and LG, will be mutually equal; becauſe the right lines DE, PB, and HI, are biſected in the ſame point, as are alſo the right lines FG, PC, and KL. Conceive now DPF, EPG to repreſent oppoſite cones deſcribed with the infinitely ſmall vertical angles DPF, EPG, and the lines DH, EI to be infinitely ſmall alſo. Then the particles of the cones DHKF, GLIE, cut off by the ſpheroidical ſuperficies, by reaſon of the equality of the lines DH and EI, will be to one another as the ſquares of the diſtances from the body P, and will therefore attract that corpuſcle equally. And by a like reaſoning if the ſpaces DPF, EGCB be divided into particles by the ſuperficies of innumerable ſimilar ſpheroids concentric to the former and having one common axis, all theſe particles will equally attract on both ſides the body

P towards contrary parts. Therefore the forces of the cone DPF, and of the conic segment EGCB, are equal, and by their contrariety destroy each other. And the case is the same of the forces of all the matter that lies without the interior spheroid PCBM. Therefore the body P is attracted by the interior spheroid PCBM alone, and therefore (by cor. 3, prop. 72) its attraction is to the force with which the body A is attracted by the whole spheroid AGOD as the distance PS to the distance AS. Q.E.D.

PROPOSITION XCII. PROBLEM XLVI.

An attracting body being given, it is required to find the ratio of the decrease of the centripetal forces tending to its several points.

The body given must be formed into a sphere, a cylinder, or some regular figure, whose law of attraction answering to any ratio of decrease may be found by prop. 80, 81, and 91. Then, by experiments, the force of the attractions must be found at several distances, and the law of attraction towards the whole, made known by that means, will give the ratio of the decrease of the forces of the several parts; which was to be found.

PROPOSITION XCIII. THEOREM XLVII.

If a solid be plane on one side, and infinitely extended on all other sides, and consist of equal particles equally attractive, whose forces decrease, in the recess from the solid, in the ratio of any power greater than the square of the distances; and a corpuscle placed towards either part of the plane is attracted by the force of the whole solid; I say, that the attractive force of the whole solid, in the recess from its plane superficies, will decrease in the ratio of a power whose side is the distance of the corpuscle from the plane, and its index less by 3 than the index of the power of the distances.

Case 1. Let LGI (Pl. 24, Fig. 6) be the plane by which the solid is terminated. Let the solid lie on that hand of the plane that is towards I, and let it be resolved into innumerable planes mHM, nIN, oKO, &c. parallel to GL. And first let the attracted body C be placed without the solid. Let there be drawn CGHI perpendicular to those innumerable planes,

and let the attractive forces of the points of the folid decreafe in the ratio of a power of the diftances whofe index is the number n not lefs than 3. Therefore (by cor. 3, prop. 90) the force with which any plane mHM attracts the point C is reciprocally as CH^{n-2}. In the plane mHM take the length HM reciprocally proportional to CH^{n-2}, and that force will be as HM. In like manner in the feveral planes lGL, nIN, oKO, &c. take the lengths GL, IN, KO, &c. reciprocally proportional to CG^{n-2}, CI^{n-2}, CK^{n-2}, &c. and the forces of thofe planes will be as the lengths fo taken, and therefore the fum of the forces as the fum of the lengths, that is, the force of the whole folid as the area GLOK produced infinitely towards OK. But that area (by the known methods of quadratures) is reciprocally as CG^{n-3}, and therefore the force of the whole folid is reciproeally as CG^{n-3}. Q.E.D.

Case 2. Let the corpufcle C (Fig. 7) be now placed on that hand of the plane lGL that is within the folid, and take the diftance CK equal to the diftance CG. And the part of the folid LGloKO terminated by the parallel planes lGL, oKO, will attract the corpufcle, fituate in the middle, neither one way nor another, the contrary actions of the oppofite points deftroying one another by reafon of their equality. Therefore the corpufcle C is attracted by the force only of the folid fituate beyond the plane OK. But this force (by cafe 1) is reciprocally as CK^{n-3}, that is (becaufe CG, CK are equal), reciprocally as CG^{n-3}. Q.E.D.

Cor. 1. Hence if the folid LGIN be terminated on each fide by two infinite parallel planes LG, IN, its attractive force is known, fubducting from the attractive force of the whole infinite folid LGKO the attractive force of the more diftant part NIKO infinitely produced towards KO.

Cor. 2. If the more diftant part of this folid be rejected, becaufe its attraction compared with the attraction of the nearer part is inconfiderable, the attraction of that nearer part will, as the diftance increafes, decreafe nearly in the ratio of the power CG^{n-3}.

Cor. 3. And hence if any finite body, plane on one fide, attract a corpufcle fituate over-againft the middle of that

plane, and the diſtance between the corpuſcle and the plane compared with the dimenſions of the attracting body be extremely ſmall; and the attracting body conſiſt of homogeneous particles, whoſe attractive forces decreaſe in the ratio of any power of the diſtances greater than the quadruplicate; the attractive force of the whole body will decreaſe very nearly in the ratio of a power whoſe ſide is that very ſmall diſtance, and the index leſs by 3 than the index of the former power. This aſſertion does not hold good, however, of a body conſiſting of particles whoſe attractive forces decreaſe in the ratio of the triplicate power of the diſtances; becauſe, in that caſe, the attraction of the remoter part of the infinite body in the ſecond corollary is always infinitely greater than the attraction of the nearer part.

SCHOLIUM.

If a body is attracted perpendicularly towards a given plane, and from the law of attraction given the motion of the body be required; the problem will be ſolved by ſeeking (by prop. 39) the motion of the body deſcending in a right line towards that plane, and (by cor. 2, of the laws) compounding that motion with an uniform motion performed in the direction of lines parallel to that plane. And, on the contrary, if there be required the law of the attraction tending towards the plane in perpendicular directions, by which the body may be cauſed to move in any given curve line, the problem will be ſolved by working after the manner of the third problem.

But the operations may be contracted by reſolving the ordinates into converging ſeries. As if to a baſe A the length B be ordinately applied in any given angle, and that length be as any power of the baſe $A^{\frac{m}{n}}$; and there be ſought the force with which a body, either attracted towards the baſe or driven from it in the direction of that ordinate, may be cauſed to move in the curve line which that ordinate always deſcribes with its ſuperior extremity; I ſuppoſe the baſe to be increaſed by a very ſmall part O, and I reſolve the ordinate $\overline{A+O}|^{\frac{m}{n}}$ into an infinite ſeries $A^{\frac{m}{n}} + \frac{m}{n} OA^{\frac{m-n}{n}} + \frac{mm-mn}{2nn}$

$OOA^{\frac{m-2n}{n}}$ &c. and I suppose the force proportional to the term of this series in which O is of two dimensions, that is, to the term $\frac{mm-mn}{2nn} OOA^{\frac{m-2n}{n}}$. Therefore the *force* sought is as $\frac{mm-mn}{nn} A^{\frac{m-2n}{n}}$, or, which is the same thing, as $\frac{mm-mn}{nn} B^{\frac{m-2n}{m}}$. As if the ordinate describe a parabola, m being $= 2$, and $n = 1$, the force will be as the given quantity $2B^{0}$, and therefore is given. Therefore with a given force the body will move in a parabola, as *Galileo* has demonstrated. If the ordinate describe an hyperbola, m being $= 0 - 1$, and $n = 1$, the force will be as $2A^{-3}$ or $2B^{3}$; and therefore a force which is as the cube of the ordinate will cause the body to move in an hyperbola. But leaving this kind of propositions, I shall go on to some others relating to motion which I have not yet touched upon.

SECTION XIV.

Of the motion of very small bodies when agitated by centripetal forces tending to the several parts of any very great body.

PROPOSITION XCIV. THEOREM XLVIII.

If two similar mediums be separated from each other by a space terminated on both sides by parallel planes, and a body in its passage through that space be attracted or impelled perpendicularly towards either of those mediums, and not agitated or hindered by any other force; and the attraction be every where the same at equal distances from either plane, taken towards the same hand of the plane; I say, that the sine of incidence upon either plane will be to the sine of emergence from the other plane in a given ratio.

CASE 1. Let Aa and Bb (Pl. 25, Fig. 1) be two parallel planes, and let the body light upon the first plane Aa in the direction of the line GH, and in its whole passage through the intermediate space let it be attracted or impelled towards the medium of incidence, and by that action let it be made to describe a curve line HI, and let it emerge in the direction of the line IK. Let there be erected IM perpendicular to Bb the

plane of emergence, and meeting the line of incidence GH prolonged in M, and the plane of incidence Aa in R; and let the line of emergence KI be produced and meet HM in L. About the centre L, with the interval LI, let a circle be described cutting both HM in P and Q, and MI produced in N; and, first, if the attraction or impulse be supposed uniform, the curve HI (by what *Galileo* has demonstrated) be a parabola, whose property is that of a rectangle under its given latus rectum and the line IM is equal to the square of HM; and moreover the line HM will be bisected in L. Whence if to MI there be let fall the perpendicular LO, MO, OR will be equal; and adding the equal lines ON, OI, the wholes MN, IR will be equal also. Therefore since IR is given, MN is also given, and the rectangle NMI is to the rectangle under the latus rectum and IM, that is, to HM^2, in a given ratio. But the rectangle NMI is equal to the rectangle PMQ, that is, to the difference of the squares ML^2, and PL^2 or LI^2; and HM^2 hath a given ratio to its fourth part ML^2; therefore the ratio of $ML^2 - LI^2$ to ML^2 is given, and by conversion the ratio of LI^2 to ML^2, and its subduplicate, the ratio of LI to ML. But in every triangle, as LMI, the sines of the angles are proportional to the opposite sides. Therefore the ratio of the sine of the angle of incidence LMR to the sine of the angle of emergence LIR is given. Q.E.D.

Case 2. Let now the body pass successively through several spaces terminated with parallel planes AabB, BbcC, &c. (Pl. 25, Fig. 2), and let it be acted on by a force which is uniform in each of them separately, but different in the different spaces; and by what was just demonstrated, the sine of the angle of incidence on the first plane Aa is to the sine of emergence from the second plane Bb in a given ratio; and this sine of incidence upon the second plane Bb will be to the sine of emergence from the third plane Cc in a given ratio; and this sine to the sine of emergence from the fourth plane Dd in a given ratio; and so on *in infinitum*; and, by equality, the sine of incidence on the first plane to the sine of emergence from the last plane in a given ratio. Let now the intervals of the planes be diminished, and their number be infinitely increased, so that the action of attraction or

impulſe, exerted according to any aſſigned law, may become continual, and the ratio of the ſine of incidence on the firſt plane to the ſine of emergence from the laſt plane being all along given, will be given then alſo. Q.E.D.

PROPOSITION XCV. THEOREM XLIX.

The ſame things being ſuppoſed, I ſay, that the velocity of the body before its incidence is to its velocity after emergence as the ſine of emergence to the ſine of incidence.

Make AH and Id equal (Pl. 25, Fig. 3), and erect the perpendiculars AG, dK meeting the lines of incidence and emergence GH, IK, in G and K. In GH take TH equal to IK, and to the plane Aa let fall a perpendicular Tv. And (by cor. 2, of the laws of motion) let the motion of the body be reſolved into two, one perpendicular to the planes Aa, Bb, Cc, &c. and another parallel to them. The force of attraction or impulſe, acting in directions perpendicular to thoſe planes, does not at all alter the motion in parallel directions; and therefore the body proceeding with this motion will in equal times go through thoſe equal parallel intervals that lie between the line AG and the point H, and between the point I and the line dK; that is, they will deſcribe the lines GH, IK in equal times. Therefore the velocity before incidence is to the velocity after emergence as GH to IK or TH, that is, as AH or Id to vH, that is (ſuppoſing TH or IK radius), as the ſine of emergence to the ſine of incidence. Q.E.D.

PROPOSITION XCVI. THEOREM L.

The ſame things being ſuppoſed, and that the motion before incidence is ſwifter than afterwards; I ſay, that if the line of incidence be inclined continually, the body will be at laſt reflected, and the angle of reflexion will be equal to the angle of incidence.

For conceive the body paſſing between the parallel planes Aa, Bb, Cc, &c. (Pl. 25, Fig. 4) to deſcribe parabolic arcs as above; and let thoſe arcs be HP, PQ, QR, &c. And let the obliquity of the line of incidence GH to the firſt plane Aa be ſuch that the ſine of incidence may be to the radius of the circle whoſe ſine it is, in the ſame ratio which the ſame ſine of in-

cidence hath to the fine of emergence from the plane Dd into the fpace DdeE; and becaufe the fine of emergence is now become equal to radius, the angle of emergence will be a right one, and therefore the line of emergence will coincide with the plane Dd. Let the body come to this plane in the point R; and becaufe the line of emergence coincides with that plane, it is manifeft that the body can proceed no farther towards the plane Ee. But neither can it proceed in the line of emergence Rd; becaufe it is perpetually attracted or impelled towards the medium of incidence. It will return, therefore, between the planes Cc, Dd, defcribing an arc of a parabola QRq, whofe principal vertex (by what *Galileo* has demonftrated) is in R, cutting the plane Cc in the fame angle at q, that it did before at Q; then going on in the parabolic arcs qp, ph, &c. fimilar and equal to the former arcs QP, PH, &c. it will cut the reft of the planes in the fame angles at p, h, &c. as it did before in P, H, &c. and will emerge at laft with the fame obliquity at h with which it firft impinged on that plane at H. Conceive now the intervals of the planes Aa, Bb, Cc, Dd, Ee, &c. to be infinitely diminifhed, and the number infinitely increafed, fo that the action of attraction or impulfe, exerted according to any affigned law, may become continual; and, the angle of emergence remaining all along equal to the angle of incidence, will be equal to the fame alfo at laft. Q.E.D.

SCHOLIUM.

Thefe attractions bear a great refemblance to the reflexions and refractions of light made in a given ratio of the fecants, as was difcovered by *Snellius*; and confequently in a given ratio of the fines, as was exhibited by *Des Cartes*. For it is now certain from the phænomena of *Jupiter*'s fatellites, confirmed by the obfervations of different aftronomers, that light is propagated in fucceffion, and requires about feven or eight minutes to travel from the fun to the earth. Moreover, the rays of light that are in our air (as lately was difcovered by *Grimaldus*, by the admiffion of light into a dark room through a fmall hole, which I have alfo tried) in their paffage near the angles of bodies, whether tranfparent or opaque (fuch as the

circular and rectangular edges of gold, silver and brass coins, or of knives, or broken pieces of stone or glass), are bent or inflected round those bodies as if they were attracted to them; and those rays which in their passage come nearest to the bodies are the most inflected, as if they were most attracted; which thing I myself have also carefully observed. And those which pass at greater distances are less inflected; and those at still greater distances are a little inflected the contrary way, and form three fringes of colours. In Pl. 25, Fig. 6, s represents the edge of a knife, or any kind of wedge AsB; and gowog, fnunf, emtme, dlsld, are rays inflected towards the knife in the arcs owo, nun, mtm, lsl; which inflection is greater or less according to their distance from the knife. Now since this inflection of the rays is performed in the air without the knife, it follows that the rays which fall upon the knife are first inflected in the air before they touch the knife. And the case is the same of the rays falling upon glass. The refraction, therefore, is made not in the point of incidence, but gradually, by a continual inflection of the rays; which is done partly in the air before they touch the glass, partly (if I mistake not) within the glass, after they have entered it; as is represented (Pl. 25, Fig. 7) in the rays ckzc, biyb, ahxa, falling upon r, q, p, and inflected between k and z, i and y, h and x. Therefore because of the analogy there is between the propagation of the rays of light and the motion of bodies, I thought it not amiss to add the following propositions for optical uses; not at all considering the nature of the rays of light, or enquiring whether they are bodies or not; but only determining the trajectories of bodies which are extremely like the trajectories of the rays.

PROPOSITION XCVII. PROBLEM XLVII.

Supposing the sine of incidence upon any superficies to be in a given ratio to the sine of emergence; and that the inflection of the paths of those bodies near that superficies is performed in a very short space, which may be considered as a point; it is required to determine such a superficies as may cause all the corpuscles issuing from any one given place to converge to another given place.

Let A (Pl. 25, Fig. 8) be the place from whence the corpuscles diverge; B the place to which they should converge; CDE the curve line which by its revolution round the axis AB describes the superficies sought; D, E, any two points of that curve; and EF, EG, perpendiculars let fall on the paths of the bodies AD, DB. Let the point D approach to and coalesce with the point E; and the ultimate ratio of the line DF by which AD is increased, to the line DG by which DB is diminished, will be the same as that of the sine of incidence to the sine of emergence. Therefore the ratio of the increment of the line AD to the decrement of the line DB is given; and therefore if in the axis AB there be taken any where the point C through which the curve CDE must pass, and CM the increment of AC be taken in that given ratio to CN the decrement of BC, and from the centres A, B, with the intervals AM, BN, there be described two circles cutting each other in D; that point D will touch the curve sought CDE, and, by touching it any where at pleasure, will determine that curve. Q.E.I.

Cor. 1. By causing the point A or B to go off sometimes *in infinitum*, and sometimes to move towards other parts of the point C, will be obtained all those figures which *Cartesius* has exhibited in his Optics and Geometry relating to refractions. The invention of which *Cartesius* having thought fit to conceal, is here laid open in this proposition.

Cor 2. If a body lighting on any superficies CD (Pl. 25, Fig. 9) in the direction of a right line AD, drawn according to any law, should emerge in the direction of another right line DK; and from the point C there be drawn curve lines CP, CQ, always perpendicular to AD, DK; the increments of the lines PD, QD, and therefore the lines themselves PD, QD, generated by those increments, will be as the sines of incidence and emergence to each other, and *è contra*.

PROPOSITION XCVIII. PROBLEM XLVIII.

The same things supposed; if round the axis AB (Pl. 25, Fig. 10) *any attractive superficies be described as* CD, *regular or irregular, through which the bodies issuing from the given place* A *must pass; it is required to find a second attractive*

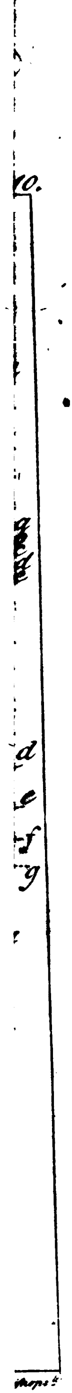
ſuperficies EF, *which may make thoſe bodies converge to a given place* B.

Let a line joining AB cut the firſt ſuperficies in C and the ſecond in E, the point D being taken any how at pleaſure. And ſuppoſing the ſine of incidence on the firſt ſuperficies to the ſine of emergence from the ſame, and the ſine of emergence from the ſecond ſuperficies to the ſine of incidence on the ſame, to be as any given quantity M to another given quantity N; then produce AB to G, ſo that BG may be to CE as M — N to N; and AD to H, ſo that AH may be equal to AG; and DF to K, ſo that DK may be to DH as N to M. Join KB, and about the centre D with the interval DH deſcribe a circle meeting KB produced in L, and draw BF parallel to DL; and the point F will touch the line EF, which, being turned round the axis AB, will deſcribe the ſuperficies ſought. Q.E.F.

For conceive the lines CP, CQ to be every where perpendicular to AD, DF, and the lines ER, ES to FB, *FD* reſpectively, and therefore QS to be always equal to CE; and (by cor. 2, prop. 97) PD will be to QD as M to N, and therefore as DL to DK, or FB to FK; and by diviſion as DL — FB or PH — PD — FB to FD or FQ — QD; and by compoſition as PH — FB to FQ, that is (becauſe PH and CG, QS and CE, are equal), as CE + BG—FR to CE — FS. But (becauſe BG is to CE as M — N to N) it comes to paſs alſo that CE + BG is to CE as M to N; and therefore, by diviſion, FR is to FS as M to N; and therefore (by cor. 2, prop. 97) the ſuperficies EF compels a body, falling upon it in the direction DF, to go on in the line FR to the place B. Q.E.D.

SCHOLIUM.

In the ſame manner one may go on to three or more ſuperficies. But of all figures the ſphærical is the moſt proper for optical uſes. If the object glaſſes of teleſcopes were made of two glaſſes of a ſphærical figure, containing water between them, it is not unlikely that the errors of the refractions made in the extreme parts of the ſuperficies of the glaſſes may be accurately enough corrected by the refractions of the water.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Such object-glaſſes are to be preferred before elliptic and hyperbolic glaſſes, not only becauſe they may be formed with more eaſe and accuracy, but becauſe the pencils of rays ſituate without the axis of the glaſs would be more accurately refracted by them. But the different refrangibility of different rays is the real obſtacle that hinders optics from being made perfect by ſphærical or any other figures. Unleſs the errors thence ariſing can be corrected, all the labour ſpent in correcting the others is quite thrown away.

END OF THE FIRST VOLUME.

INDEX TO VOL. I.

ERRATUM.

At the end of the Note in page 117 add the following words;—"Printed in the 4th Volume of the Manchester Memoirs."

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